Online Semidefinite Programming for Power System State Estimation

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***Abstract*—Power system state estimation (PSSE) constitutes a crucial prerequisite for reliable operation of the power grid. A key challenge for accurate PSSE is the inherent nonlinearity of SCADA measurements in the system states. Recent proposals for static PSSE tackle this issue by exploiting hidden convexity structure and solving a semidefinite programming (SDP) relaxation. In this work, an *online* PSSE algorithm based on SDP relaxation is proposed, which enjoys a similar convexity advantage, while capitalizing on past measurements as well for improved performance. An online convex optimization technique is adopted to derive an efficient algorithm with strong performance guarantees. Numerical tests verify the efficacy of the proposed approach.**

I. Introduction

Power system state estimation (PSSE) is crucial for reliable operation of the power grid. Based on various types of measurements collected at different buses and lines across the grid, PSSE aims at recovering unknown system states, which are the complex voltages at all buses in the network. Accurate PSSE is critical for detecting instabilities and contingencies, preventing massive blackout events that can be caused by cascade failure. The importance of PSSE will be amplified in the future smart grid, where volatility due to renewable resources as well as aggravated demand due to transportation electrification will pose significant challenges to the grid monitoring, management and control [1].

In order to perform this key task successfully, installation of phasor measurement units (PMUs) has been advocated. PMUs can acquire GPS-synchronized voltage measurements at a much higher rate than the existing SCADA system. However, as PMUs are deployed progressively, it is still essential to incorporate SCADA measurements for redundancy and robustness of PSSE.

One of the challenges in PSSE involving SCADA measurements is that the measurements are nonlinear in the system states. Traditional approaches adopt a weighted nonlinear least-squares formulation, typically solved by Newton-type numerical methods [2]. However, since the problem is inherently nonconvex, there is no guarantee that the solution found is globally optimal. In fact, depending on the initial point, the iterative solver may get stuck at a locally optimal solution. This is particularly problematic in the challenging scenarios, where the system states change significantly between measurements, or the measurements become corrupted by bad data, leaving fewer available measurements.

Recent proposals mitigate this issue by exploiting hidden convexity structure of the power flow relations [3], [4], [5]. Specifically, by lifting the state vector to a higher-dimensional space involving symmetric positive semidefinite matrices, one can formulate a semidefinite programming (SDP) problem, which is convex and whose solution often leads to the globally optimal solution of the original PSSE problem.

Traditionally, the PSSE problem has been often solved in a static setup, which ignores the dynamics of power systems, and thus does not exploit the past observations. *Dynamic* PSSE, on the other hand, can enjoy improved reliability, robustness and observability, even under bad data and topology errors, as well as achieve state predictability [6], [7]. Faster measurement updates owing to PMUs facilitate dynamic PSSE, while various smart grid challenges make the approach imperative.

Similar to the static PSSE, dynamic PSSE has to deal with nonlinearity in measurements. Various nonlinear filtering techniques such as the extended Kalman filter and the unscented Kalman filter have been employed to address this issue [7], [8]. However, such approximate nonlinear filtering algorithms may suffer under severe nonlinear dynamics. Moreover, estimation of the possibly non-stationary dynamics of the system is in itself a challenging issue [6], [9].

In this work, we do not explicitly commit to a model of the power system state dynamics, but still capitalize on the accumulated measurements. An instrumental framework for this purpose is online convex optimization (OCO), which has been recently popular for real-time machine learning applications [10]. In conjunction with convex relaxation of the nonlinear PSSE problem, the framework naturally provides an online PSSE algorithm, with strong performance guarantees.

The rest of the paper is organized as follows. In Section II, the static PSSE formulation as well as the SDP relaxation approach are reviewed. In Section III the OCO framework is introduced and an online PSSE algorithm is developed. Numerical tests are presented in Section IV. Conclusions are provided in Section V.

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II. REVIEW OF STATIC POWER SYSTEM STATE ESTIMATION

In this section, the basic formulation of the static PSSE problem is presented, and a solution technique based on SDP relaxation recently proposed in [5] is reviewed to lay the groundwork for our online formulation in Sec. III.

A. Static PSSE Problem Statement

Let $\mathcal{N} := \{1, 2, \ldots, N\}$ denote the set of buses in a transmission network consisting of N buses, and let $\mathcal{L} := \{(n, n') : n, n' \in \mathcal{N}\}$ represent the transmission lines in the network. The static PSSE problem is to obtain the complex voltage estimates of all buses in a power network for a given time based on the measurements acquired at a subset of the buses and lines. Observations on various power system quantities include: i) the active and reactive powers injected at bus $n \in \mathcal{N}$, denoted as P_n and Q_n, respectively; ii) the active and reactive power flows from bus n to n' for $(n, n') \in \mathcal{L}$, denoted as $P_{nn'}$ and $Q_{nn'}$, respectively; and iii) the voltage magnitude $|V_n|$ at bus $n \in \mathcal{N}$. Collectively, denote such M measurements as $\mathbf{z} := [z_1, z_2, \ldots, z_M]^{\mathcal{T}}$, where z_m corresponds to the m-th measurement regardless of its type, and $\cdot^{\mathcal{T}}$ stands for transposition. Given $\mathbf{z}$, the goal of PSSE is to obtain $\mathbf{v} := [V_1, V_2, \ldots, V_N]^{\mathcal{T}}$, or V_n at all $n \in \mathcal{N}$.

The measurements $\mathbf{z}$ are nonlinearly related to the unknown $\mathbf{v}$. Specifically, let $\mathbf{Y} \in \mathbb{C}^{N \times N}$ be the admittance matrix of the transmission grid, and $\mathbf{i} := [i_1, i_2, \ldots, i_N]^{\mathcal{T}}$ the vector of injected currents at all buses. Then, Kirchhoff's and Ohm's laws yield

$$\mathbf{i} = \mathbf{Y}\mathbf{v} \tag{1}$$

and the complex power injected to node $n \in \mathcal{N}$ is given by

$$P_n + jQ_n = V_n I_n^* \tag{2}$$

where $\cdot^*$ denotes complex conjugation. Similarly, the line currents $I_{nn'}$ for $(n, n') \in \mathcal{L}$ are given by

$$I_{nn'} = y_{s,nn'} V_n + y_{nn'}(V_n - V_{n'}) \tag{3}$$

where $y_{s,nn'}$ represents the shunt admittance at bus n associated with line (n, n'). The complex power flow over line (n, n') is then given as

$$P_{nn'} + jQ_{nn'} = V_n I_{nn'}^*. \tag{4}$$

Encapsulating the generally linear-quadratic relation between z_m and $\mathbf{v}$ through $h_m(\cdot)$, one obtains the measurement model

$$z_m = h_m(\mathbf{v}) + \epsilon_m, \quad m = 1, 2, \ldots, M \tag{5}$$

where ϵ_m is assumed to be zero-mean Gaussian with variance σ_m^2, and independent across m. Under this model, the static PSSE problem is formulated as a weighted nonlinear least-squares one, given by

$$\min_{\mathbf{v}} \sum_{m=1}^{M} w_m \left[z_m - h_m(\mathbf{v})\right]^2 \tag{6}$$

where $w_m := (\sigma_m^2)^{-1}$.

B. SDP Relaxation Approach

Problem (6) is nonconvex, and various numerical methods such as Newton's iteration have been employed targeting the local optima of this problem [2]. Recently, an SDP relaxation-based approach showed much potential to locate a globally optimal solution. The key observation is that upon defining $\mathbf{x} := [\Re(\mathbf{v})^{\mathcal{T}} \Im(\mathbf{v})^{\mathcal{T}}]^{\mathcal{T}}$, which stacks the real and imaginary parts of vector $\mathbf{v}$, and letting $\mathbf{X} := \mathbf{x}\mathbf{x}^{\mathcal{T}}$, one can express measurement function $h_m(\mathbf{v})$ as linear in $\mathbf{X}$; that is,

$$h_m(\mathbf{v}) = \mathrm{tr}(\mathbf{H}_m \mathbf{X}) \tag{7}$$

for some $\mathbf{H}_m \in \mathbb{R}^{2N \times 2N}$, which is a matrix that may depend on the underlying line parameters. Substituting (7) into (6), one obtains an equivalent formulation given by

$$\min_{\mathbf{X}} \sum_{m=1}^{M} w_m \left[z_m - \mathrm{tr}(\mathbf{H}_m \mathbf{X})\right]^2 \tag{8a}$$

$$\text{subject to } \mathbf{X} \succeq \mathbf{0} \tag{8b}$$

$$\mathrm{rank}(\mathbf{X}) = 1. \tag{8c}$$

Problem (8) is nonconvex due to the rank constraint (8c). Removing (8c) renders the problem (8a)–(8b) convex. This, in turn, can be shown (using Schur's complement lemma) to be equivalent to an SDP given next, which can be solved very efficiently [11].

$$\min_{\mathbf{X}, \boldsymbol{\alpha}} \mathbf{w}^{\mathcal{T}} \boldsymbol{\alpha} \tag{9a}$$

$$\text{subject to } \begin{bmatrix} -\alpha_m & z_m - \mathrm{tr}(\mathbf{H}_m \mathbf{X}) \\ z_m - \mathrm{tr}(\mathbf{H}_m \mathbf{X}) & -1 \end{bmatrix} \preceq \mathbf{0}, \quad m = 1, \ldots, M \tag{9b}$$

$$\mathbf{X} \succeq \mathbf{0} \tag{9c}$$

where $\boldsymbol{\alpha} := [\alpha_1, \alpha_2, \ldots, \alpha_M]^{\mathcal{T}}$ and $\mathbf{w} := [w_1, w_2, \ldots, w_M]^{\mathcal{T}}$. If the optimal solution $\mathbf{X}_{\mathrm{opt}}$ for (9) has rank 1 such that $\mathbf{X}_{\mathrm{opt}} = \mathbf{x}_{\mathrm{opt}} \mathbf{x}_{\mathrm{opt}}^{\mathcal{T}}$, then $\mathbf{x}_{\mathrm{opt}} = [\Re(\mathbf{v}_{\mathrm{opt}})^{\mathcal{T}}, \Im(\mathbf{v}_{\mathrm{opt}})^{\mathcal{T}}]^{\mathcal{T}}$ yields the globally optimal solution $\mathbf{v}_{\mathrm{opt}}$ for (6) and $\alpha_{m,\mathrm{opt}}^2 = [z_m - h_m(\mathbf{v}_{\mathrm{opt}})]^2$. If $\mathrm{rank}(\mathbf{X}_{\mathrm{opt}}) > 1$, the best rank-1 approximation $\sqrt{\lambda_1}\mathbf{u}_1$ can be used, where λ_1 and $\mathbf{u}_1$ are the largest eigenvalue and the corresponding unit-norm eigenvector of $\mathbf{X}_{\mathrm{opt}}$, respectively.

III. ONLINE POWER SYSTEM STATE ESTIMATION

Online PSSE aims at capitalizing on the entire measurement history to obtain state estimates of improved accuracy at manageable complexity, while at the same time track slow variation emerging due to time-varying line parameters, load or renewable generations, which contain uncertainty. Dynamic state-space models have been considered along with nonlinear filtering techniques such as the extended and the unscented Kalman filters [6], [7], [8]. In this work, we advocate the use of the online convex optimization (OCO) framework [10], which does not require an explicit dynamical model, but still provides guaranteed performance. A brief overview of the OCO framework is first given next, followed by its application to an online PSSE formulation.

A. Online Convex Optimization Models

The OCO model considers a multi-stage game between a player and an adversary. (In the present setup, the utility that performs PSSE assumes the role of the player, while the load and the renewable generators can be regarded as the adversary.) At time t, the player chooses an action $\mathbf{X}^t \in \mathcal{X}$, and subsequently the adversary reveals a convex function $c^t : \mathcal{X} \to \mathbb{R}$. As will be specified in the next subsection, $\mathbf{X}^t$ will be the rank-relaxed matrix variable for (8a)–(8b), and c^t the quadratic fitting cost. Then, the player suffers the loss of amount $c^t(\mathbf{X}^t)$. The goal of the player is to minimize the so-called *regret* $R_c(T)$ over T stages, defined as

$$R_c(T) := \sum_{t=1}^{T} c^t(\mathbf{X}^t) - \min_{\mathbf{X}\in\mathcal{X}} \sum_{t=1}^{T} c^t(\mathbf{X}) \tag{10}$$

which corresponds to the cost incurred by the player relative to the cost due to a single best action $\mathbf{X}^* := \arg\min_{\mathbf{X}\in\mathcal{X}} \sum_{t=1}^{T} c^t(\mathbf{X})$, which is selected with the benefit of knowing c^t for all $t = 1, 2, \ldots, T$ in hindsight. Under appropriate conditions, online iterative algorithms can be constructed, which yield $\mathbf{X}^t$ at time t, to achieve the regret upper-bounds that grow sublinearly in T; i.e., $R_c(T)/T \to 0$ as $T \to \infty$. This means that the online algorithms can eventually perform as well as the fixed action selected in hindsight in terms of per-stage cost.

B. Online PSSE Algorithm

The online PSSE problem can be stated as follows. At time $t \in \{1, 2, \ldots, T\}$, a set of M measurements $\mathbf{z}^t := [z_1^t, z_2^t, \ldots, z_M^t]^{\mathcal{T}}$ is obtained. Based on the entire history of measurements $\{\mathbf{z}^\tau\}_{\tau=1}^t$, the state vector $\mathbf{v}^t := [v_1^t, \ldots, v_N^t]^{\mathcal{T}}$ must be estimated.

To solve this problem without explicit modeling of the dynamics of $\{\mathbf{z}^t\}$, the OCO approach is taken to solve (8a)–(8b). That is, upon defining

$$c^t(\mathbf{X}) := \sum_{m=1}^{M} w_m [z_m^t - \mathrm{tr}(\mathbf{H}_m \mathbf{X})]^2 \tag{11}$$

the goal is to choose $\mathbf{X}^t \succeq 0$ at each time $t \in \{1, 2, \ldots, T\}$ so as to minimize the total cost $\sum_{t=1}^{T} c^t(\mathbf{X}^t)$. The power system state $\mathbf{v}^t$ at time t can be recovered from $\mathbf{X}^t$ as described in Sec. II-B.

A widely used OCO algorithm is the online mirror descent (OMD) method, which is a projected (sub)gradient method with a proximal term based on the Bregman divergence [12]. The OMD iterations constitute an efficient first-order algorithm with sublinear convergence rate [13].

Specifically, $\mathbf{X}^{t+1} \in \mathbb{R}^{2N\times 2N}$ is obtained in OMD as

$$\mathbf{X}^{t+1} = \arg\min_{\mathbf{X}\succeq 0} \langle \nabla c^t(\mathbf{X}^t), \mathbf{X}\rangle + \frac{1}{\eta^t} D(\mathbf{X}, \mathbf{X}^t) \tag{12}$$

where η^t denotes a step size, and $D(\cdot,\cdot)$ represents a Bregman divergence, which encourages the search to be done in the proximity of the current iterate $\mathbf{X}^t$. There are a number of Bregman divergences that can be used for matrices, including the Frobenius, von Neumann and log-det divergences [14]. In this work, we adopt the Frobenius divergence given by

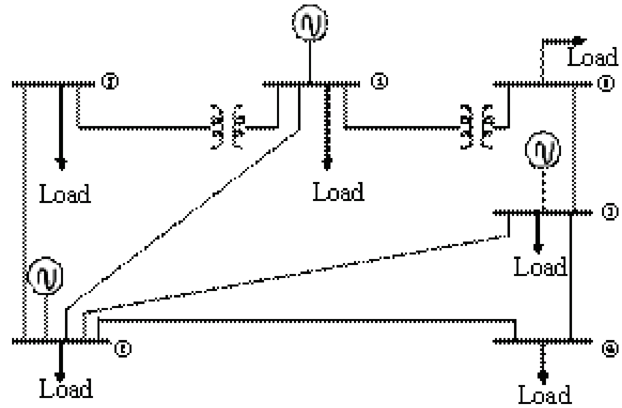

Fig. 1. IEEE 6-bus test system.

$$D(\mathbf{X}, \mathbf{Y}) = \frac{1}{2} ||\mathbf{X} - \mathbf{Y}||_F^2. \tag{13}$$

Substituting (13) and (11) into (12) yields

$$\mathbf{X}^{t+1} = \arg\max_{\mathbf{X}\succeq 0} \left\{ \sum_{m=1}^{M} 2w_m [z_m^t - \mathrm{tr}(\mathbf{H}_m \mathbf{X}^t)] \mathrm{tr}(\mathbf{H}_m \mathbf{X}) + \frac{1}{2\eta^t} ||\mathbf{X} - \mathbf{X}^t||_F^2 \right\} \tag{14}$$

which can be solved efficiently as well [11].

IV. Numerical Tests

The proposed online PSSE algorithm was tested using the IEEE 6-bus test system with 11 lines, which is depicted in Fig. 1. In order to compare the performance of the proposed algorithm, the nonlinear weighted least-squares (WLS) problem (6) was solved. For this, a Matlab toolbox called MATPOWER [15] was used to generate the pertinent power flow and all meter measurements corresponding to this 6-bus system. Then, the state estimation function DoSE in MATPOWER was utilized to estimate the power state in each time slot. For our online PSSE update in (14), a convex optimization package CVX [11], together with its embedded interior-point solver SeDuMi [16], was used.

To simulate slow dynamics of the system states, $\mathbf{x}^t$ were varied according to a first-order autoregressive model given by $\mathbf{v}^{t+1} = \rho\mathbf{v}^t + \nu^t$, where each entry of ν^t had a zero-mean Gaussian magnitude of variance 0.05^2 and an angle sampled from uniform distribution in $[-0.08\pi, 0.08\pi]$. Then, $\mathbf{x}^t := [\Re(\mathbf{v}^t)^{\mathcal{T}}\ \Im(\mathbf{v}^t)^{\mathcal{T}}]^{\mathcal{T}}$. The value of ρ was set to 0.95. The initial state $\mathbf{x}^0$ was randomly generated with each bus's magnitude Gaussian with mean 1 and variance 0.01^2 and the angle uniformly distributed over $[-0.5\pi, 0.5\pi]$. For the sake of fixing the phase angle ambiguity, bus 1 was initialized as a reference bus with magnitude 1 and angle 0.

The active and reactive power flows across all 11 lines as well as the voltage magnitudes at all 6 buses were measured. All measurements are corrupted with mutually independent additive Gaussian noise with variances $\sigma_m^2 = 0.02^2$. The WLS estimator was initialized with a flat voltage profile, namely,

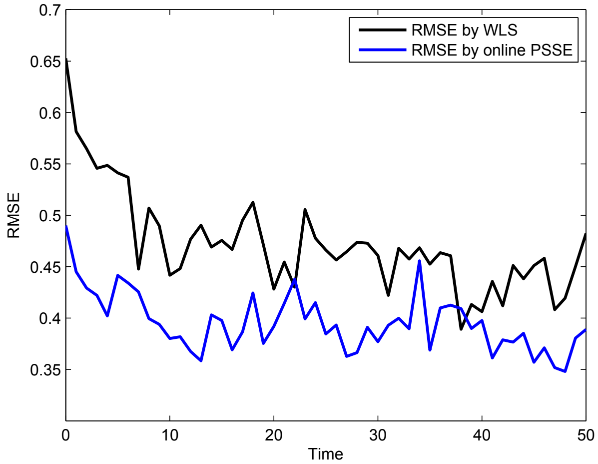

Fig. 2. RMSE performances.

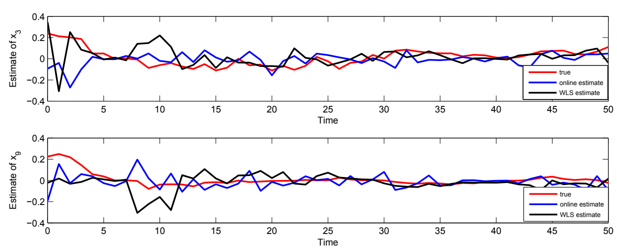

Fig. 3. Evolution of the real and the imaginary parts of v_3^t

with the all-one vector. The proposed algorithm was initialized by an SDP-based solution to (9).

Fig. 2 compares the root-mean-square-errors (RMSEs) of online PSSE and the WLS-based estimates, where the averaging was done over 50 independent realizations. Fig. 3 shows the evolution of the estimates calculated by the two methods as well as the true state v_3^t of bus 3. The top panel corresponds to the real part of v_3^t and the bottom the imaginary part. It can be seen that sometimes the WLS estimate induces large error, while our SDP-based method stays close to the true state.

V. Conclusion

An online PSSE algorithm has been proposed, which improves upon existing static PSSE by exploiting the past as well as present measurements. To mitigate the inherent nonlinearity and nonconvexity of the PSSE problem, an SDP relaxation approach was adopted, which performs the search in a higher dimensional space of symmetric positive semidefinite matrices. An OCO technique was applied to yield an efficient projected gradient descent algorithm. The proposed algorithm was tested in the IEEE 6-bus test system, and performance advantages were found compared to conventional approaches.

References

[1] G. B. Giannakis, V. Kekatos, N. Gatsis, S.-J. Kim, H. Zhu, and B. F. Wollenberg, "Monitoring and optimization for power grids," *IEEE Sig. Proc. Mag.*, vol. 30, no. 5, pp. 107–128, Sep. 2013.

[2] A. Monticelli, "Electric power system state estimation," *Proc. IEEE*, vol. 88, no. 2, pp. 262–282, Feb. 2000.

[3] X. Bai, H. Wei, K. Fujisawa, and Y. Wang, "Semidefinite programming for optimal power flow problems," *Intl. J. Electr. Power & Energy Syst.*, vol. 30, no. 6–7, pp. 383–392, Jul. 2008.

[4] J. Lavaei and S. H. Low, "Zero duality gap in optimal power flow problem," *IEEE Trans. Power Syst.*, vol. 27, no. 1, pp. 92–107, Feb. 2012.

[5] H. Zhu and G. B. Giannakis, "Estimating the state of AC power systems using semidefinite programming," in *Proc. of the North American Power Symposium*, Boston, MA, Aug. 2011, pp. 1–7.

[6] P. Rousseaux and Th. Van Cutsem, "Whither dynamic state estimation?" *Electrical Power & Energy Systems*, vol. 12, no. 2, pp. 104–116, Apr. 1990.

[7] S.-J. Huang and K.-R. Shih, "Dynamic-state-estimation scheme including nonlinear measurement-function considerations," *IEE Proc.–Gener. Transm. Distrib.*, vol. 149, no. 6, pp. 673–678, Nov. 2002.

[8] S. Wang, W. Gao, and A. P. Meliopoulos, "An alternative method for power system dynamic state estimation based on unscented transform," *IEEE Trans. Power Syst.*, vol. 27, no. 2, pp. 942–950, May 2012.

[9] Y. Weng, R. Negi, and M. D. Ilić, "Historical data-driven state estimation for electric power systems," in *Proc. of the IEEE SmartGridComm Symposium*, Vancouver, Canada, Oct. 2013, pp. 97–102.

[10] S. Shalev-Shwartz, "Online learning and online convex optimization," *Foundations and Trends in Machine Learning*, vol. 4, no. 2, pp. 107–194, Mar. 2012.

[11] L. Vandenberghe and S. Boyd, "Semidefinite programming," *SIAM Rev.*, vol. 38, no. 1, pp. 49–95, Mar. 1996.

[12] A. Beck and M. Teboulle, "Mirror descent and nonlinear projected subgradient methods for convex optimization," *Oper. Res. Lett.*, vol. 31, no. 3, pp. 167–175, 2003.

[13] N. Srebro, K. Sridharan, and A. Tewari, "On the universality of online mirror descent," in *Proc. of Advances in Neural Information Processing Systems*, Granada, Spain, Dec. 2011.

[14] B. Kulis, M. A. Sustik, and I. S. Dhillon, "Low-rank kernel learning with Bregman matrix divergences," *J. Mach. Learn. Res.*, vol. 10, pp. 341–376, Feb. 2009.

[15] R. D. Zimmerman, C. E. Murillo-Sánchez, and R. J. Thomas, "Matpower: Steady-state operations, planning, and analysis tools for power systems research and education," *IEEE Trans. Power Syst.*, vol. 26, no. 1, pp. 12–19, 2011.

[16] J. F. Sturm, "Using sedumi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optimization methods and software*, vol. 11, no. 1-4, pp. 625–653, 1999.