Online Power System State Estimation Using Alternating Direction Method of Multipliers

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***Abstract*—The convex relaxation approaches for power system state estimation (PSSE) offer robust alternatives to the conventional PSSE algorithms, by avoiding local optima and providing guaranteed convergence, critical especially when the states deviate significantly from the nominal conditions. On the other hand, the associated semidefinite programming problem may be computationally demanding. In this work, a variable splitting technique called alternating direction method of multipliers is employed to reduce the complexity, and also efficiently accommodate a regularizer promoting desired low-rank matrix solutions. Both static and online formulations are developed. Numerical tests verify the efficacy of the proposed techniques.**

I. Introduction

The goal of power system state estimation (PSSE) is to estimate the voltages at the individual buses in the grid from various types of measurements collected from a subset of the buses and the lines in the grid. Accurate PSSE must be warranted even when there are significant deviations in the states from the nominal operating conditions, in order to detect instabilities and prevent cascade failures [1].

The challenges for PSSE in the future power system evolution is the high variability of the states anticipated owing to the incorporation of volatile renewable energy sources, as well as larger load such as plug-in electric vehicles. The opportunities, on the other hand, include the availability of synchro-phasor measurements, whose samples are collected at a far higher rate than the existing SCADA system with precise time stamps. However, the deployment of phasor measurement units (PMUs) is progressive, and accurate PSSE still mandates capitalizing on the SCADA measurements.

One of the challenges in incorporating SCADA data is that they are highly nonlinear in the states. Traditional approaches adopt a weighted least-squares (WLS) formulation, which is inherently nonconvex. Gauss-Newton-type solvers are often sensitive to initialization, falling into local optima, and even suffer from divergence. This is particularly problematic in challenging scenarios, such as when the system states vary significantly between measurements, or when the available measurements are few, possibly due to bad data corruption.

Recent studies show that a hidden convexity structure in the power flow relations can be exploited to mitigate these issues [2], [3], [4], [5]. The idea is to lift the voltage variables to a space of positive semidefinite matrices, where a relaxed convex semidefinite programming (SDP) problem can be formulated. Its solution can then be used to obtain a near-optimal solution to the original problem in many practical instances.

The SDP problem, albeit convex, is still computationally demanding especially for large-scale grids. Our contribution in this work is to devise a low-complexity iterative algorithm using a variable splitting technique called alternating direction method of multipliers (ADMM) [6]. To facilitate the recovery of the voltage estimates from the matrix variable, a nuclear norm-based penalty is augmented to the WLS cost, which encourages low-rank matrix solutions. The ADMM technique can handle this feature without additional computational cost.

Static PSSE ignores the system state dynamics, and does not leverage the past measurements. Dynamic PSSE recognizes this opportunity to provide improved reliability, robustness, and observability, and even predictions on the future states. Similar to the static counterpart, dynamic PSSE needs to deal with the nonlinearity. Various nonlinear filtering techniques such as the extended Kalman filter and the unscented Kalman filter have been employed [7], as well as the SDP relaxation idea [8]. However, these techniques are computationally complex, and require precise models on the system dynamics.

The online learning framework offers a promising alternative, as it obviates elaborate models for the dynamics, while still capitalizing on the past data. The convex relaxation approach was married to the online convex optimization (OCO) technique in [9] to develop an online PSSE algorithm with provable convergence guarantees. The present work permeates the benefits of ADMM to the online setting as well, to devise a low-complexity alternative to the SDP-based online PSSE.

The remainder of the paper is organized as follows. In Sec. II, the static PSSE formulations are reviewed and the ADMM-based algorithm is developed. In Sec. III, the online PSSE algorithm using online ADMM is derived. The performances of the proposed algorithms are verified via numerical tests in Sec. IV. Conclusions are provided in Sec. V.

II. Static PSSE Using ADMM

A. Static PSSE Problem

The PSSE aims at estimating the complex voltages at all buses in the network, given various types of measurements taken at a subset of buses and lines. Let $\mathcal{N} := \{1, 2, \ldots, N\}$ denote the set of N buses, and $\mathcal{E}$ the lines. Let $\mathcal{N}_n := \{n' :$

$(n,n') \in \mathcal{E}\}$ be the set of neighboring buses of bus n. Let vector $\mathbf{v} := [V_1, V_2, \ldots, V_N]^{\mathcal{T}} \in \mathbb{C}^N$ denote the voltages $\{V_n\}_{n\in\mathcal{N}}$ at all buses, and $\mathbf{i} := [I_1, I_2, \ldots, I_N]^{\mathcal{T}}$ the injection currents $\{I_n\}_{n\in\mathcal{N}}$, where ${}^{\mathcal{T}}$ represents the transposition.

Possible measurements for PSSE include: the active and the reactive powers injected at bus $n \in \mathcal{N}$, denoted by P_n and Q_n, respectively; the active and reactive power flows from bus n to bus n' over line $(n,n') \in \mathcal{E}$, denoted by $P_{nn'}$ and $Q_{nn'}$, respectively; and the voltage magnitudes $|V_n|$ at bus $n \in \mathcal{N}$. One can collect these measurements in a vector $\mathbf{z} := [z_1, z_2, \ldots, z_M]^{\mathcal{T}} \in \mathbb{R}^M$, where M is the total number of measurements. The goal of PSSE is to estimate $\mathbf{v}$ given $\mathbf{z}$.

For each line $(n,n') \in \mathcal{E}$, the admittance is denoted as $y_{nn'} = g_{nn'} + jb_{nn'}$, where $j := \sqrt{-1}$, and the admittance matrix $\mathbf{Y} \in \mathbb{C}^{N\times N}$ is defined by setting its (n,n')-entries as

$$Y_{nn'} = \begin{cases} \sum_{n'\in\mathcal{N}_n} y_{nn'}, & \text{if } n = n' \\ -y_{nn'}, & \text{if } n \neq n',\ (n,n') \in \mathcal{E} \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Then, Kirchhoff's law and Ohm's law yield

$$I_n = \sum_{n'\in\mathcal{N}_n} (V_n - V_{n'}) y_{nn'} \tag{2}$$

which can be written compactly as $\mathbf{i} = \mathbf{Y}\mathbf{v}$. The complex power injected to node $n \in \mathcal{N}$ is given by

$$S_n := P_n + jQ_n = V_n I_n^* \tag{3}$$

where P_n and Q_n represent the real and the reactive powers injected to bus n, respectively, and * denotes complex conjugation. Likewise, the line currents $I_{nn'}$ for $(n,n') \in \mathcal{E}$ are given by

$$I_{nn'} = y_{s,nn'} V_n + y_{nn'}(V_n - V_{n'}) \tag{4}$$

where $y_{s,nn'}$ is the shunt admittance at bus n associated with line (n,n'). The complex power flow $S_{nn'}$ over line (n,n') can then be expressed as

$$S_{nn'} := P_{nn'} + jQ_{nn'} = V_n I_{nn'}^*. \tag{5}$$

In view of (3) and (5), the relevant measurements are related nonlinearly (in fact, quadratically) to the variables of interest. Encapsulating these nonlinear relationship between z_m and $\mathbf{v}$ through $h_m(\cdot)$, one can postulate the measurement model as

$$z_m = h_m(\mathbf{v}) + \epsilon_m, \quad m = 1, 2, \ldots, M \tag{6}$$

where ϵ_m is zero-mean white Gaussian noise with variance σ_m^2. Based on this model, the PSSE problem is formulated as a nonlinear WLS one as

$$\min_{\mathbf{v}} \sum_{m=1}^{M} \frac{1}{\sigma_m^2} \left[z_m - h_m(\mathbf{v})\right]^2. \tag{7}$$

Problem (7) is nonconvex, and thus it is typically hard to obtain the globally optimal solutions. Various numerical methods such as the Gauss-Newton iteration target the local optima of the problem [1].

B. Convex Relaxation Approach to PSSE

Recently, convex relaxation approaches showed much potential in finding globally optimal solutions to OPF [2], [3], as well as PSSE problems [4]. The key observation is that upon defining $\mathbf{X} := \mathbf{x}\mathbf{x}^H$, with ^{H} denoting Hermitian transposition, one can express $\{h_m(\mathbf{v})\}$ linearly in $\mathbf{X}$; i.e.,

$$h_m(\mathbf{v}) = \langle \mathbf{H}_m, \mathbf{X} \rangle = \mathrm{tr}\{\mathbf{H}_m^H \mathbf{X}\} \tag{8}$$

for some Hermitian matrix $\mathbf{H}_m \in \mathbb{C}^{N\times N}$, where $\langle \cdot, \cdot \rangle$ denotes the matrix inner product. To see this, let $\mathbf{e}_n$ denote the n-th canonical vector in $\mathbb{R}^N$. Then, upon defining

$$\mathbf{Y}_n := \mathbf{e}_n \mathbf{e}_n^{\mathcal{T}} \mathbf{Y} \tag{9}$$

$$\mathbf{Y}_{nn'} := (y_{s,nn'} + y_{nn'})\mathbf{e}_n \mathbf{e}_n^{\mathcal{T}} - y_{nn'} \mathbf{e}_n \mathbf{e}_{n'}^{\mathcal{T}} \tag{10}$$

it can be verified that

$$S_n = \mathrm{tr}\{\mathbf{Y}_n^H \mathbf{X}\} \tag{11}$$

$$S_{nn'} = \mathrm{tr}\{\mathbf{Y}_{nn'}^H \mathbf{X}\} \tag{12}$$

$$|V_n|^2 = \mathrm{tr}\{\mathbf{e}_n \mathbf{e}_n^{\mathcal{T}} \mathbf{X}\}. \tag{13}$$

Thus, by defining

$$\bar{\mathbf{Y}}_n := \frac{1}{2}\left(\mathbf{Y}_n + \mathbf{Y}_n^H\right),\ \bar{\mathbf{Y}}_{nn'} := \frac{1}{2}\left(\mathbf{Y}_{nn'} + \mathbf{Y}_{nn'}^H\right)$$
$$\tilde{\mathbf{Y}}_n := \frac{j}{2}\left(\mathbf{Y}_n - \mathbf{Y}_n^H\right),\ \tilde{\mathbf{Y}}_{nn'} := \frac{j}{2}\left(\mathbf{Y}_{nn'} - \mathbf{Y}_{nn'}^H\right) \tag{14}$$

one can identify

$$P_n = \mathrm{tr}\{\bar{\mathbf{Y}}_n^H \mathbf{X}\},\ Q_n = \mathrm{tr}\{\tilde{\mathbf{Y}}_n^H \mathbf{X}\} \tag{15}$$

$$P_{nn'} = \mathrm{tr}\{\bar{\mathbf{Y}}_{nn'}^H \mathbf{X}\},\ Q_{nn'} = \mathrm{tr}\{\tilde{\mathbf{Y}}_{nn'}^H \mathbf{X}\}. \tag{16}$$

Therefore, one can rewrite (7) equivalently as

$$\min_{\mathbf{X}\succeq\mathbf{0}} \sum_{m=1}^{M} \frac{1}{\sigma_m^2} \left[z_m - \mathrm{tr}\{\mathbf{H}_m^H \mathbf{X}\}\right]^2 \tag{17}$$

$$\text{subject to } \mathrm{rank}\{\mathbf{X}\} = 1. \tag{18}$$

Problem (17)–(18) would be convex, were it not for the rank constraint (18). Convex relaxation amounts to simply neglecting (18), and solving just (17), which can be reformulated as a SDP problem. If the solution to the relaxed problem has rank equal to 1, it is clear that the solution is also the minimizer for the original problem (17)–(18), and the voltage estimate $\hat{\mathbf{v}}$ can be easily recovered from eigendecomposition.

For convex relaxation approaches for the OPF problem, much research has been devoted to show that the relaxed problem indeed yields rank-1 solutions for various practical cases. For PSSE, the relaxed problem typically does not yield a rank-1 solution, although a low-rank solution is still expected [4]. Hence, heuristics are employed to obtain $\hat{\mathbf{v}}$. Typical methods include picking the dominant eigenvector of $\mathbf{X}$, or taking multiple samples from a Gaussian distribution with mean $\mathbf{0}$ and covariance $\mathbf{X}$, and choosing the one (with appropriate scaling) that best fits the measurements [4].

C. ADMM-Based Solution

In statistical learning, the prior knowledge of low-rank matrix solutions is often encoded to the problem by using a nuclear norm-based constraint. The nuclear norm $\|\mathbf{X}\|_*$ of matrix $\mathbf{X}$ is the sum of the absolute values of the singular values of $\mathbf{X}$. It is advocated frequently as a convex surrogate of the matrix rank [10]. Specifically, one can consider the following variation of (17)–(18).

$$\min_{\mathbf{X}\succeq\mathbf{0}} \sum_{m=1}^{M} \frac{1}{\sigma_m^2}\left[z_m - \mathrm{tr}\{\mathbf{H}_m^H\mathbf{X}\}\right]^2 + \lambda\|\mathbf{X}\|_* \tag{19}$$

where $\lambda \geq 0$ is the weight to adjust the severity of the regularization. Note that for sufficiently large λ, $\mathbf{X} = \mathbf{0}$ becomes the solution to (19), while $\lambda = 0$ recovers the relaxed PSSE problem (17). By setting λ appropriately, the solution to (19) can be forced to have rank close to 1.

Problem (19) is convex and can be solved via standard convex optimization software. However, generic optimization algorithms tend to scale poorly as the problem size is increased as they do not exploit the inherent structure of the PSSE problem. In this work, a variable splitting technique called ADMM is employed to reduce the computational complexity [6]. By using ADMM, one can reduce (19) into a series of unconstrained quadratic programs (QPs) and eigendecompositions.

First, by introducing an auxiliary variable $\mathbf{Z} \in \mathbb{C}^{N\times N}$, the following problem equivalent to (19) is constructed.

$$\min_{\mathbf{X}=\mathbf{X}^H,\mathbf{Z}\succeq\mathbf{0}} \sum_{m=1}^{M} \frac{1}{\sigma_m^2}\left[z_m - \mathrm{tr}\{\mathbf{H}_m^H\mathbf{X}\}\right]^2 + \lambda\|\mathbf{Z}\|_* \tag{20a}$$

$$\text{subject to } \mathbf{X} = \mathbf{Z}. \tag{20b}$$

To apply the ADMM, an augmented Lagrangian is formed as

$$L(\mathbf{X},\mathbf{Z};\mathbf{\Lambda}) := \sum_{m=1}^{M} \frac{1}{\sigma_m^2}\left[z_m - \mathrm{tr}\{\mathbf{H}_m^H\mathbf{X}\}\right]^2 + \lambda\|\mathbf{Z}\|_* + \langle\mathbf{\Lambda},\mathbf{X}-\mathbf{Z}\rangle + \frac{\rho}{2}\|\mathbf{X}-\mathbf{Z}\|_F^2 \tag{21}$$

where $\|\cdot\|_F^2$ denotes the Frobenius norm, $\rho > 0$ is a positive constant, and $\mathbf{\Lambda}$ is the Lagrange multiplier matrix associated with constraint (20b). Then, the following updates with iteration index i, repeated until convergence, can be shown to yield an optimal solution to (20).

$$\mathbf{X}(i+1) = \arg\min_{\mathbf{X}=\mathbf{X}^H} L(\mathbf{X},\mathbf{Z}(i);\mathbf{\Lambda}(i)) \tag{22a}$$

$$\mathbf{Z}(i+1) = \arg\min_{\mathbf{Z}\succeq 0} L(\mathbf{X}(i+1),\mathbf{Z};\mathbf{\Lambda}(i)) \tag{22b}$$

$$\mathbf{\Lambda}(i+1) = \mathbf{\Lambda}(i) + \rho\left[\mathbf{X}(i+1) - \mathbf{Z}(i+1)\right] \tag{22c}$$

It is noted that the update for $\mathbf{X}$ in (22a) can be transformed to an unconstrained QP involving N^2 real variables, whose solution can be found by setting the derivative to zero and solving the corresponding system of linear equations. On the other hand, the update for $\mathbf{Z}$ in (22b) can be re-written as

$$\mathbf{Z}(i+1) = \arg\min_{\mathbf{Z}\succeq 0} \frac{1}{2}\|\mathbf{Z} - \tilde{\mathbf{X}}(i)\|_F^2 + \frac{\lambda}{\rho}\|\mathbf{Z}\|_* \tag{23}$$

where $\tilde{\mathbf{X}}(i) := \mathbf{X}(i+1) + \rho^{-1}\mathbf{\Lambda}(i)$. Thus, with the eigendecomposition of $\tilde{\mathbf{X}}(i)$ given as $\tilde{\mathbf{X}}(i) = \mathbf{Q}(i)\mathbf{D}(i)\mathbf{Q}(i)^H$ with unitary $\mathbf{Q}(i)$ and diagonal $\mathbf{D}(i)$, $\mathbf{Z}(i+1)$ is given simply by

$$\mathbf{Z}(i+1) = \mathbf{Q}(i)\left(\mathbf{D}(i) - \frac{\lambda}{\rho}\mathbf{I}\right)^+ \mathbf{Q}(i)^H \tag{24}$$

where $(\cdot)^+ := \max\{0,\cdot\}$ applies entry-wise.

III. Online PSSE

An online PSSE formulation based on the OCO framework was proposed in [9], where at each iteration, a SDP problem was solved. In the present work, the benefits of ADMM is permeated to this online setup.

A. Online convex optimization approach

The OCO model considers a multi-stage game between a player and an adversary [11]. In the present setup, the utility that performs PSSE assumes the role of the player, while the grid can be regarded as the adversary. At time t, the player chooses an action (estimate) $\mathbf{X}^t \in \mathcal{X}$, and subsequently the adversary reveals a convex function $c^t : \mathcal{X} \to \mathbb{R}$. As was formulated in (19), $\mathbf{X}^t$ will be the rank-relaxed matrix variable, and c^t the fitting cost with any regularizers. Then, the player suffers a loss of amount $c^t(\mathbf{X}^t)$. The goal of the player is to minimize the so-called *regret*, defined over T stages as

$$R(T) := \sum_{t=1}^{T} c^t(\mathbf{X}^t) - \min_{\mathbf{X}\in\mathcal{X}} \sum_{t=1}^{T} c^t(\mathbf{X}) \tag{25}$$

which corresponds to the cumulative cost incurred by the online plays, relative to the cost due to a single best action $\mathbf{X}^* := \arg\min_{\mathbf{X}\in\mathcal{X}} \sum_{t=1}^{T} c^t(\mathbf{X})$, selected in hindsight with the advantage of knowing $\{c^t\}$ for all $t = 1, 2, \ldots, T$.

Under appropriate conditions, algorithms can be constructed to yield $\mathbf{X}^t$ at each time slot t, based on the data available up to time t, which achieve a regret that grows sublinearly in T; i.e., $R(T)/T \to 0$ as $T \to \infty$. This means that the online algorithm can perform as well as the fixed best action in terms of the per-stage cost. Remarkably, such a guarantee is obtained without any assumptions on the dynamics of the adversary. In fact, the adversary can even be strategic in providing $\{c^t\}$, trying to beat the purpose of the player, which speaks to the robustness of the framework.

B. Online PSSE Using ADMM

The online PSSE utilizes past measurements as well as the current ones to improve the estimation accuracy. It can also track temporal variations in the states due to the time-varying load and volatile renewable generation. Formally, the online PSSE problem can be stated as follows. At time slot $t \in \{1, 2, \ldots, T\}$, based on the measurements collected so far, i.e., $\{z_m^\tau\}_{\tau=1}^t$, estimate $\{V_n^t\}$ for all $n \in \mathcal{N}$, to achieve a regret $R(T)$ that is sublinear in T.

To address this problem without explicitly modeling the state dynamics, the OCO approach is pursued. Consider the cost function at time slot t given by

$$c^t(\mathbf{X}) := \phi^t(\mathbf{X}) + r(\mathbf{X}) \tag{26}$$

with

$$\phi^t(\mathbf{X}) := \sum_{m=1}^{M} \frac{1}{\sigma_m^2} \left[z_m^t - \mathrm{tr}\{\mathbf{H}_m^H \mathbf{X}\}\right]^2 \tag{27}$$

$$r(\mathbf{X}) := \lambda \|\mathbf{X}\|_*. \tag{28}$$

An OCO algorithm with sublinear regret can be derived for this setup using a variety of methods, such as the composite objective online mirror descent (COMID) or the regularized dual averaging (RDA) methods [12], [13]. However, such an approach would end up producing the update rules that involve solving SDP problems. One may apply the ADMM technique for solving the SDP at each time t, but this leads to a double-loop algorithm, where the inner loop corresponds to the ADMM iterations in i, and the outer loop in time slots t.

To take advantage of the ADMM framework and avoid a double-loop implementation, a promising idea is to run the ADMM update only once at each time slot t, based on the online ADMM approach [14]. The online ADMM aims at solving

$$\min_{\mathbf{X}=\mathbf{X}^H, \mathbf{Z}\succeq \mathbf{0}} \sum_{t=1}^{T} \left[\phi^t(\mathbf{X}) + r(\mathbf{Z})\right] \tag{29}$$

$$\text{subject to } \mathbf{X} = \mathbf{Z} \tag{30}$$

in an online fashion. The augmented Lagrangian is formed as [cf. (21)]

$$L^t(\mathbf{X}, \mathbf{Z}; \mathbf{\Lambda}) := \phi^t(\mathbf{X}) + r(\mathbf{Z}) + \langle \mathbf{\Lambda}, \mathbf{X} - \mathbf{Z}\rangle + \frac{\rho}{2}\|\mathbf{X} - \mathbf{Z}\|_F^2 \tag{31}$$

from which the update rules are derived as

$$\begin{aligned}
\mathbf{X}^{t+1} &= \arg\min_{\mathbf{X}=\mathbf{X}^H} L^t(\mathbf{X}, \mathbf{Z}^t; \mathbf{\Lambda}^t) \\
&= \arg\min_{\mathbf{X}=\mathbf{X}^H} \phi^t(\mathbf{X}) + \langle \mathbf{\Lambda}^t, \mathbf{X}\rangle + \frac{\rho}{2}\|\mathbf{X} - \mathbf{Z}^t\|_F^2 && \text{(32a)} \\
\mathbf{Z}^{t+1} &= \arg\min_{\mathbf{Z}\succeq 0} L^t(\mathbf{X}^{t+1}, \mathbf{Z}; \mathbf{\Lambda}^t) \\
&= \mathbf{Q}^t \left(\mathbf{D}^t - \frac{\lambda}{\rho}\mathbf{I}\right)^+ (\mathbf{Q}^t)^H && \text{(32b)} \\
\mathbf{\Lambda}^{t+1} &= \mathbf{\Lambda}^t + \rho[\mathbf{X}^{t+1} - \mathbf{Z}^{t+1}] && \text{(32c)}
\end{aligned}$$

where $\tilde{\mathbf{X}}^t := \mathbf{X}^{t+1} + \rho^{-1}\mathbf{\Lambda}^t$ admits an eigendecomposition $\tilde{\mathbf{X}}^t = \mathbf{Q}^t \mathbf{D}^t (\mathbf{Q}^t)^H$.

Note that the update rules in (32) are essentially the same as (22), except that (22) iterates using the same set of measurements, while for (32), a new set of measurements are processed in each iteration. Obviously, with $z_m^t = z_m$ for all t and m, the online update (32) becomes equivalent to the static update in (22).

Under appropriate regularity conditions, it can be shown that the iterates $\{\mathbf{Z}^t\}$ achieves a sublinear regret bound; i.e.,

$$R(T) = \sum_{t=1}^{T} c^t(\mathbf{Z}^t) - \min_{\mathbf{X}\succeq 0} \sum_{t=1}^{T} c^t(\mathbf{X}) = o(T). \tag{33}$$

Thus, the desired voltage estimates $\hat{\mathbf{v}}^t$ at each time slot t are extracted from $\mathbf{Z}^t$, say by picking the dominant eigenvector.

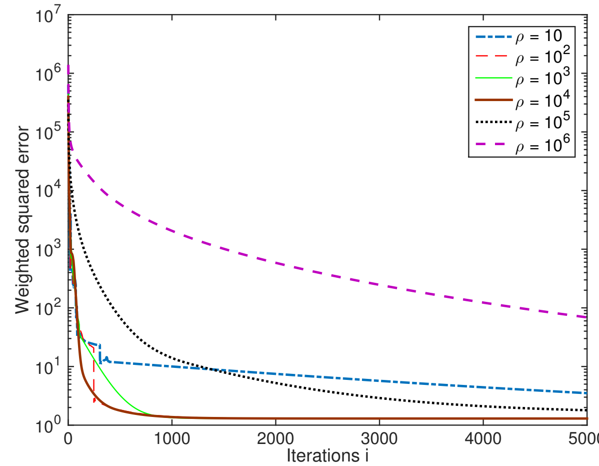

Fig. 1. Convergence of SDP/ADMM algorithm.

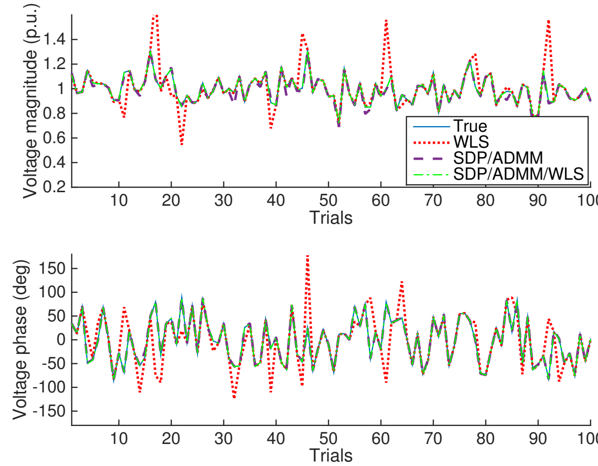

Fig. 2. True and estimated voltages at bus 2.

IV. Numerical Tests

To see the effectiveness of the proposed methods, numerical tests were performed. The IEEE 9-bus test system with 9 lines from [15] was used. The real and the reactive power flows over the 9 lines, measured at one end of each line, and the voltage magnitudes at all buses, were employed as measurements. The variance of the measurement noise for the power flows was set to 4×10^{-4} and that for the voltage magnitudes to 1×10^{-4}.

To generate the true voltage profile for testing the proposed static PSSE, the voltage magnitudes were sampled from a Gaussian distribution with mean 1 and variance 0.01, and the phases from a uniform distribution with support $[-\theta, \theta]$ for a given $\theta > 0$. Fig. 1 shows the convergence of the weighted squared error, which is the objective in (17), as the number of ADMM iterations i grows, for different values of ρ. The value of θ was set to $\pi/2$. It is seen that the algorithm essentially converges in 1,000 iterations when ρ is chosen to be $\rho = 10^4$.

Fig. 2 depicts the true and the estimated voltages at bus 2

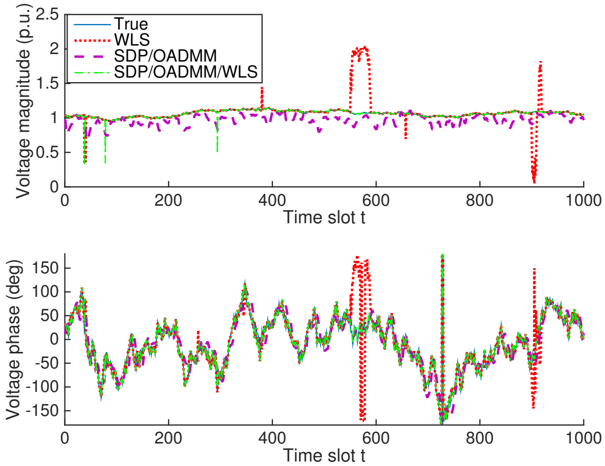

Fig. 3. True and estimated voltages at bus 3.

for 100 different trials. The top panel in Fig. 2 represents the magnitudes, and the bottom panel the phases. The solid curves depict the true values, and the estimates from the proposed SDP/ADMM algorithm are plotted in the dashed ones. A hundred samples from Gaussian distribution with the scaling as suggested in [4] were used to choose the final estimates. For comparison, the estimates from the conventional Gauss-Newton-based WLS solution, initialized with the flat voltage profile (i.e., $V_n = 1 \; \forall n$), are also plotted as the dotted curves. Finally, the dash-dot curves, termed as SDP/ADMM/WLS in Fig. 2, represent the estimates from the WLS algorithm initialized with the SDP/ADMM estimates. The value of θ was again set to $\pi/2$ and $\lambda = 0$ was used. It can be seen that the WLS estimates often deviate significantly from the true values, perhaps due to convergence to local optima, but the proposed algorithms effectively avoid it.

The root-mean-square-error (RMSE) $\sqrt{\|\mathbf{v} - \hat{\mathbf{v}}\|^2}$ obtained from 500 independent trials, and also averaged over the buses, were observed to be 0.596, 0.0300 and 0.0108 for WLS, SDP/ADMM, and SDP/ADMM/WLS, respectively, which corroborates the superior performance of the proposed methods to the conventional one. When λ was set to 10^2, the RMSE for SDP/ADMM was decreased to 0.0262, which highlights the benefit of including the nuclear-norm regularizer. However, in this particular setup, the corresponding RMSE due to SDP/ADMM/WLS did not change.

To test the online algorithms, time-varying states were generated. A first-order autoregressive (AR) model with the AR coefficient $\alpha_{\text{mag}} = 0.99999$ was used for the magnitude variation, and coefficient $\alpha_{\text{pha}} = 0.98$ for the phase. Fig. 3 shows the true and the estimated voltage magnitudes for $T = 1{,}000$ time slots. The online ADMM-based algorithm described in Section III-B was employed with $\lambda = \rho = 10^3$. The dominant eigenvector was chosen as the voltage estimate. The WLS algorithm was initialized by the previous time slot's estimate to improve the performance upon the flat voltage profile-based initialization. It is seen that the proposed online ADMM-based algorithm does not fall into the local optima as the WLS method, although this time there is some bias in the magnitude estimate. The RMSE averaged over the last 900 time slots was equal to 0.196 for the WLS, while it was 0.251 for the online ADMM method. However, when the SDP/OADMM estimate was fed to the WLS method to further refine the estimate, the RMSE dropped to 0.0191, which is ten times better than that from the plain WLS method.

V. Conclusion

The SDP relaxation-based PSSE problem was tackled using the ADMM technique to reduce the computational complexity and to incorporate a low-rank-promoting nuclear norm-based regularizer. Both the static and online formulations were derived. Compared to solving a SDP problem, the proposed technique involves solving a system of linear equations for N^2 real variables and an eigendecomposition of an N-by-N Hermitian matrix at each iteration. The numerical tests verified that the proposed SDP/ADMM techniques avoid local optima, and further polishing using WLS initialized with the SDP/ADMM estimates provides excellent performance.

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