Linear Algebra Review

CMSC 435/634

Abstract Vectors

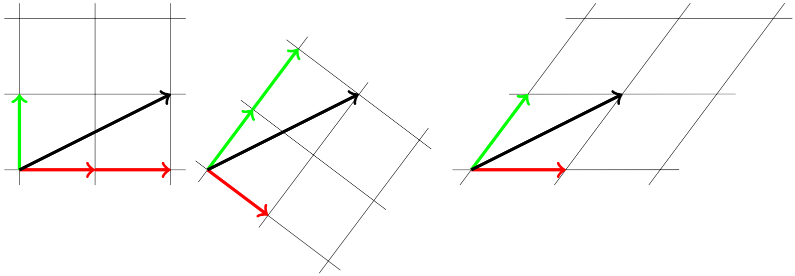
($\vec{u}$, $\vec{v}$, $\vec{w}$ vectors; a, b, c scalars)

- *Addition*: $\vec{u} + \vec{v}$ is a vector
- *Scalar Multiplication*: $a\vec{u}$ is a vector
- *Commutitive*: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- *Distributive*: $(a + b)\vec{u} = a\vec{u} + b\vec{u}$
- *Distributive*: $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$
- *Associative*: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

Basis Vectors

Vector as linear combination of *basis vectors*

- $\vec{v} = 2\hat{i} + 1\hat{j} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
- $\vec{v} = 1\hat{m} + 2\hat{n} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
- $\vec{v} = 1\hat{p} + 1\hat{q} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



Notation

- Column: $\vec{v} = \begin{bmatrix} v_0 \\ v_1 \end{bmatrix}$
  - Some texts use columns for everything
- Row: $\vec{v} = \begin{bmatrix} v_0 & v_1 \end{bmatrix}$
  - Some texts use rows for everything
  - Results in transposes and swapped order from what we'll use
- I like columns for points/vectors, rows for normals

Matrices

- Matrix: $A = \begin{bmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{bmatrix} = [a_{i,j}] = [a_{row,column}]$
- Transpose: $A^T = \begin{bmatrix} a_{0,0} & a_{1,0} \\ a_{0,1} & a_{1,1} \end{bmatrix} = [a_{j,i}]$
- Multiply: $AB = \begin{bmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{bmatrix} \begin{bmatrix} b_{0,0} & b_{0,1} \\ b_{1,0} & b_{1,1} \end{bmatrix} =$
$$\begin{bmatrix} a_{0,0}b_{0,0} + a_{0,1}b_{1,0} & a_{0,0}b_{0,1} + a_{0,1}b_{1,1} \\ a_{1,0}b_{0,0} + a_{1,1}b_{1,0} & a_{1,0}b_{0,1} + a_{1,1}b_{1,1} \end{bmatrix}$$

Matrix Code

- Math: $C = A\ B$
- Components: $c_{i,j} = \sum_\alpha a_{i,\alpha}\ b_{\alpha,j}$
- Code:

```
for(int i=0; i<N; ++i) {
  for(int j=0; j<M; ++j) {
    c[i][j] = 0;
    for(int α=0; α<K; α++) {
      c[i][j] = c[i][j] + a[i][α] * b[α][j];
    }
  }
}
```

Adjugate and Inverse

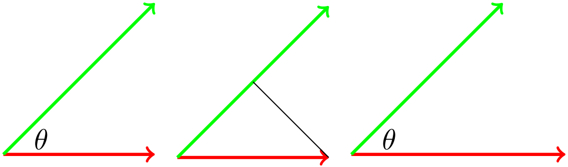
- Inverse: $A^{-1}A = AA^{-1} = I$
- Determinant: $|A|$
  - $|a| = a$
  - $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a|d| - b|c|$
  - $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$
- Adjugate: $A^* = cof(A)^T$ (matrix of cofactors $cof(A)$)
  - Sometimes called Adjoint or Adjunct
- $A^{-1} = \frac{A^*}{|A|}$

Dot Product

- Also called inner product
  - $\vec{u} \bullet \vec{v}$ is a scalar
  - *Commutitive*: $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$
  - *Distributive*: $(a\vec{u}) \bullet \vec{v} = \vec{u} \bullet (a\vec{v}) = a(\vec{u} \bullet \vec{v})$
  - *Associative*: $(\vec{u} + \vec{v}) \bullet \vec{w} = \vec{u} \bullet \vec{w} + \vec{v} \bullet \vec{w}$
  - $\vec{v} \bullet \vec{v} \geq 0$
  - $\vec{v} \bullet \vec{v} = 0 \leftrightarrow \vec{v} = \vec{0}$
- Equivalent notations
  - Vector: $\vec{u} \bullet \vec{v}$
  - Matrix: $U^T V$
  - $\sum_\alpha u_\alpha v_\alpha$

Dot Defines Length and Angle

- $\vec{v} \bullet \vec{v} = |\vec{v}|^2$
- $\vec{u} \bullet \vec{v} = |\vec{u}||\vec{v}| \cos\theta$
    - **Defines** angle θ!
    - If $|\vec{v}| = 1$, gives projection of $\vec{u}$ onto $\vec{v}$
    - If $|\vec{u}| = |\vec{v}| = 1$, gives just $\cos\theta$



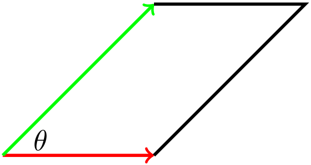
Orthogonal & Normal

- *Orthogonal* = perpendicular: $\vec{u} \bullet \vec{v} = 0$
- *Normal* (this usage) = unit-length: $\vec{u} \bullet \vec{u} = 1$
- *Orthonormal*: set of vectors both orthogonal and normal
- *Orthogonal matrix*: rows (& columns) **orthonormal**
  - For orthogonal matrices, $A^{-1} = A^T$

3D Cross Product

$\vec{u} \times \vec{v}$

- length = area of parallelogram = twice area of triangle
  - $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin(\theta)$
- direction = perpendicular to $\vec{u}$ and $\vec{v}$ (right hand rule)
- $\vec{w} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & & \\ \hat{j} & U & V \\ \hat{k} & & \end{vmatrix}$
- $\begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} u_1 v_2 - u_2 v_1 \\ u_2 v_0 - u_0 v_2 \\ u_0 v_1 - u_1 v_0 \end{bmatrix}$
- Positive terms follow 012012 order
- Netative terms follow 210210 order



Building an Orthogonal Basis

Vectors $\vec{u}$, $\vec{v}$, $\vec{w}$

- Gram-Schmidt (any number of dimensions)
  - $\vec{u'} = \vec{u}$
  - $\vec{v'} = \vec{v} - \hat{u}' \left(\vec{v} \bullet \hat{u}'\right)$
  - $\vec{v'} = \vec{v} - \frac{\vec{u'}}{|\vec{u'}|} \left(\vec{v} \bullet \frac{\vec{u'}}{|\vec{u'}|}\right)$
  - $\vec{v'} = \vec{v} - \vec{u'}\ \vec{v} \bullet \vec{u'} / |\vec{u'}|^2$
  - $\vec{v'} = \vec{v} - \vec{u'}\ \vec{v} \bullet \vec{u'} / \vec{u'} \bullet \vec{u'}$
  - $\vec{w'} = \vec{w} - \vec{u'}\ \vec{w} \bullet \vec{u'} / \vec{u'} \bullet \vec{u'} - \vec{v'}\ \vec{w} \bullet \vec{v'} / \vec{v'} \bullet \vec{v'}$
- Cross-product (3D only)
  - $\vec{u'} = \vec{u}$
  - $\vec{w'} = \vec{u'} \times \vec{v}$
  - $\vec{v'} = \vec{w'} \times \vec{u'}$