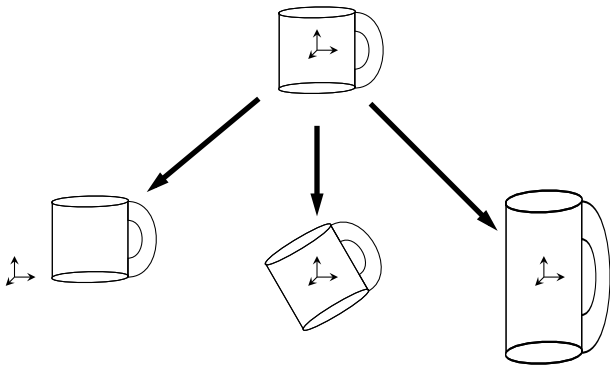


3D Transformations

CMSC 435/634

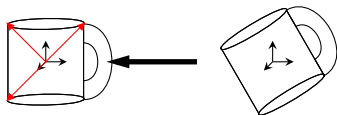
Transformation

Webster: The operation of changing one configuration or expression into another in accordance with a mathematical rule



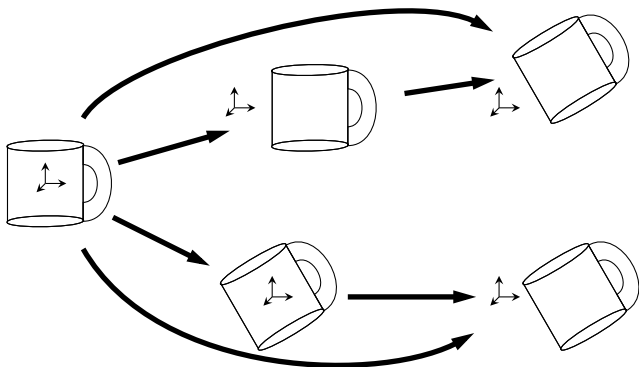
Using Transformation

- Points on object represented as vector offset from origin
- Transform is a vector to vector function
 - $\vec{p}' = f(\vec{p})$
- Relativity:
 - From \vec{p}' point of view, object is transformed
 - From \vec{p} point of view, coordinate system changes
- Inverse transform, $\vec{p} = f^{-1}(\vec{p}')$



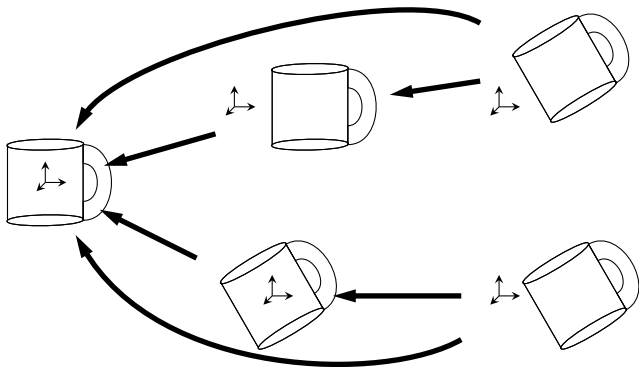
Composing Transforms

- Order matters
 - $R(T(\vec{p})) = R \circ T(\vec{p})$
 - $T(R(\vec{p})) = T \circ R(\vec{p})$



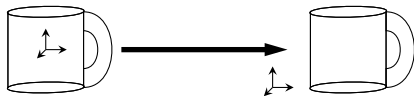
Inverting Composed Transforms

- Reverse order
 - $(R \circ T)^{-1}(\vec{p}') = T^{-1}(R^{-1}(\vec{p}'))$
 - $(T \circ R)^{-1}(\vec{p}') = R^{-1}(T^{-1}(\vec{p}'))$



Translation

- $\vec{p}' = \vec{p} + \vec{t}$
- $$\begin{bmatrix} p'^x \\ p'^y \\ p'^z \end{bmatrix} = \begin{bmatrix} p^x \\ p^y \\ p^z \end{bmatrix} + \begin{bmatrix} t^x \\ t^y \\ t^z \end{bmatrix} = \begin{bmatrix} p^x + t^x \\ p^y + t^y \\ p^z + t^z \end{bmatrix}$$
- \vec{t} says where \vec{p} -space origin ends up ($\vec{p}' = \vec{0} + \vec{t}$)
- Composition: $\vec{p}' = (\vec{p} + \vec{t}_0) + \vec{t}_1 = \vec{p} + (\vec{t}_0 + \vec{t}_1)$



Linear Transforms

- $$\begin{bmatrix} p'^x \\ p'^y \\ p'^z \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} p^x \\ p^y \\ p^z \end{bmatrix}$$

- Matrix says where \vec{p} -space axes end up

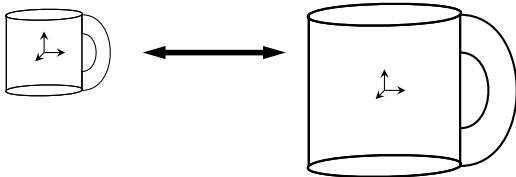
- $$\begin{bmatrix} a \\ d \\ g \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
- $$\begin{bmatrix} b \\ e \\ h \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
- $$\begin{bmatrix} c \\ f \\ i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Composition: $\vec{p}' = M (N \vec{p}) = (M N) \vec{p}$

Common case: Scaling

- $$\begin{bmatrix} p'^x \\ p'^y \\ p'^z \end{bmatrix} = \begin{bmatrix} s_x p^x \\ s_y p^y \\ s_z p^z \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix} \begin{bmatrix} p^x \\ p^y \\ p^z \end{bmatrix}$$

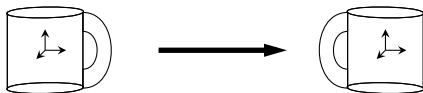
- Inverse:
$$\begin{bmatrix} 1/s_x & 0 & 0 \\ 0 & 1/s_y & 0 \\ 0 & 0 & 1/s_z \end{bmatrix}$$



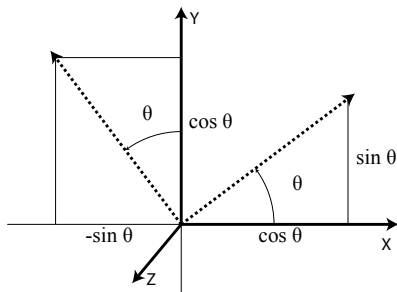
Common case: Reflection

- Negative scaling

- $$\begin{bmatrix} p'^x \\ p'^y \\ p'^z \end{bmatrix} = \begin{bmatrix} -p^x \\ p^y \\ p^z \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p^x \\ p^y \\ p^z \end{bmatrix}$$



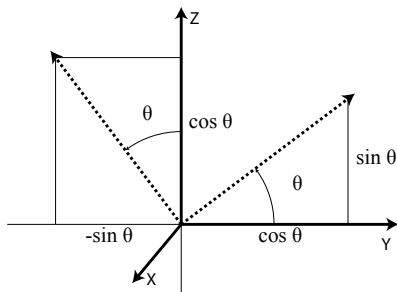
Common case: Rotation



- Orthogonal, so $M^{-1} = M^T$

- Rotate around Z: $\vec{p}' = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{p}$

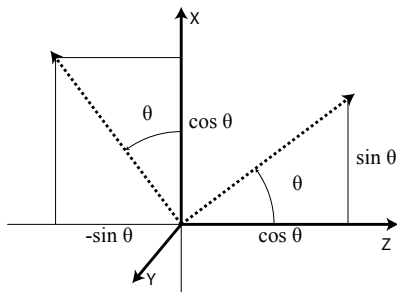
Common case: Rotation



- Orthogonal, so $M^{-1} = M^T$

- Rotate around X: $\vec{p}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \vec{p}$

Common case: Rotation



- Orthogonal, so $M^{-1} = M^T$

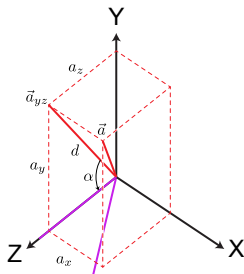
- Rotate around Y: $\vec{p}' = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \vec{p}$

Composing Transforms

- Scale by s along axis \vec{a}
 - Rotate to align \vec{a} with Z
 - Scale along Z
 - Rotate back

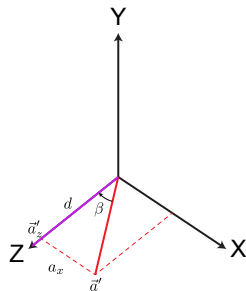
Rotate by α around X into XZ plane

- Projection of \vec{a} onto YZ: $\vec{a}_{yz} = \begin{bmatrix} 0 \\ a^y \\ a^z \end{bmatrix}$
- length $d = \sqrt{(a^y)^2 + (a^z)^2}$
- So $\cos \alpha = a^z/d$, $\sin \alpha = a^y/d$
- $R_X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a^z/d & -a^y/d \\ 0 & a^y/d & a^z/d \end{bmatrix}$
- Result $\vec{a}' = \begin{bmatrix} a^x \\ 0 \\ d \end{bmatrix}$



Rotate by $-\beta$ around Y to Z axis

- $\vec{a}' = \begin{bmatrix} a^x \\ 0 \\ d \end{bmatrix}$
- length = 1
- So $\cos \beta = d$, $\sin \beta = a^x$
- $R_Y = \begin{bmatrix} d & 0 & -a^x \\ 0 & 1 & 0 \\ a^x & 0 & d \end{bmatrix}$
- Result $\vec{a}'' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$



Composing Transforms

- Scale by s along Z : $S_Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{bmatrix}$
- Scale by s along axis \vec{a}
 - Rotate to align \vec{a} with Z
 - Scale along Z
 - Rotate back
 - $\vec{p}' = R_X^{-1} R_Y^{-1} S_Z R_Y R_X \vec{p}$

Affine Transforms

- Affine = Linear + Translation
- Composition? $A (B \vec{p} + \vec{t}_0) + \vec{t}_1 = A B \vec{p} + A \vec{t}_0 + \vec{t}_1$
- Yuck!

Homogeneous Coordinates

- Add a '1' to each point

- $$\begin{bmatrix} p^x \\ p^y \\ p^z \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} a & b & c & t^x \\ d & e & f & t^y \\ g & h & i & t^z \\ \hline 0 & 0 & 0 & 1 \end{array} \right] \begin{bmatrix} p^x \\ p^y \\ p^z \\ 1 \end{bmatrix}$$

- $\vec{p}'^x = (a p^x + b p^y + c p^z) + t^x$
- $\vec{p}'^y = (d p^x + e p^y + f p^z) + t^y$
- $\vec{p}'^z = (g p^x + h p^y + i p^z) + t^z$
- $1 = (0p^x + 0p^y + 0p^z) + 1$

Homogeneous Coordinates

- $$\begin{bmatrix} p'^x \\ p'^y \\ p'^z \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} a & b & c & t^x \\ d & e & f & t^y \\ g & h & i & t^z \\ \hline 0 & 0 & 0 & 1 \end{array} \right] \begin{bmatrix} p^x \\ p^y \\ p^z \\ 1 \end{bmatrix}$$
- $$\vec{p}' = [\vec{x} \quad \vec{y} \quad \vec{z} \mid \vec{t}] \vec{p}$$
 - \vec{t} says where the \vec{p} -space origin ends up
 - $\vec{x}, \vec{y}, \vec{z}$ say where the \vec{p} -space axes end up
- Composition: Just matrix multiplies!

Composing Transforms

- Rotate by θ about line between \vec{p}_0 and \vec{p}_1 :
 - Translate \vec{p}_0 to origin
 - Rotate to align $\vec{p}_1 - \vec{p}_0$ with Z
 - Rotate by θ around Z
 - Undo $\vec{p}_1 - \vec{p}_0$ rotation
 - Undo translation
- $T^{-1}R_X^{-1}R_Y^{-1}R_Z(\theta)R_YR_XT$

Vectors

- Transform by *Jacobian Matrix*
- Matrix of partial derivatives

$$\bullet \begin{bmatrix} \vec{p}'^x \\ \vec{p}'^y \\ \vec{p}'^z \end{bmatrix} = \begin{bmatrix} a p^x + b p^y + c p^z + t^x \\ d p^x + e p^y + f p^z + t^y \\ g p^x + h p^y + i p^z + t^z \end{bmatrix}$$

$$\bullet J = \begin{bmatrix} \partial p'^x / \partial p^x & \partial p'^x / \partial p^y & \partial p'^x / \partial p^z \\ \partial p'^y / \partial p^x & \partial p'^y / \partial p^y & \partial p'^y / \partial p^z \\ \partial p'^z / \partial p^x & \partial p'^z / \partial p^y & \partial p'^z / \partial p^z \end{bmatrix}$$

$$\bullet J = \begin{bmatrix} a & b & c \\ c & d & f \\ g & h & i \end{bmatrix}$$

- *Upper-left 3x3*

Normals

- Normal should remain perpendicular to tangent vector

- $\vec{n} \cdot \vec{v} = \vec{n}' \cdot \vec{v}' = 0$

- $$\begin{bmatrix} n_x & n_y & n_z \end{bmatrix} \begin{bmatrix} v^x \\ v^y \\ v^z \end{bmatrix} = \left(\begin{bmatrix} n_x & n_y & n_z \end{bmatrix} J^{-1} \right) \left(J \begin{bmatrix} v^x \\ v^y \\ v^z \end{bmatrix} \right) = 0$$

- $\vec{n}' = \vec{n} J^{-1}$

- Multiply by inverse on right

- OR multiply *column* normal by inverse transpose

- $(J^{-1})^T = J$ if J is orthogonal (only rotations)