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THE SECOND HOMOTOPY GROUP OF A SPUN KNOT

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§1. INTRODUCTION

IN [1] Andrews and Lomonaco calculated the second homotopy group of spun knots as $J\pi_1$-modules. In particular, they proved the following theorem:

THEOREM 2. *If $k(S^2) \subset S^4$ is a 2-sphere formed by spinning an arc A about the sphere S^2 and $(x_0, x_1, \ldots, x_n : r_1, \ldots, r_m)$ is a presentation of $\pi_1(S^4 - k(S^2))$ with x_0 the image of the generator of $\pi_1(S^2 - A)$ under the inclusion map, then*

$$(x_i(1 \leqslant i \leqslant n) \colon \textstyle\sum_{i=1}^{n} (\partial r_j/\partial x_i)\, x_i = 0.\ (1 \leqslant j \leqslant m))$$

is a presentation of $\pi_2(S^4 - k(S^2))$ as a $J\pi_1$-module.

In their proof of this theorem Fox's free derivatives suddenly and unaccountably appear in a tedious combinatorial argument. This argument, however, does not give an insight as to why they occur, a fact which strongly suggests that there is a more direct approach to the calculation.

In this paper we take a totally different approach via Reidemeister homotopy chains, which gives a simpler and more natural proof enabling us to visualize geometrically the free derivatives. Briefly the proof is as follows: We collapse the complement $C - A$ of the arc A in C to a 2-dimensional CW-complex K, where C is the closure of the component of $S^3 - S^2$ containing A. Then it follows from [5] that the matrix of the boundary homomorphism $\partial_2 : C_2(\tilde{K}) \to C_1(\tilde{K})$ is simply Fox's Jacobian matrix $(\partial r_i/\partial x_j)$ of free derivatives, where $\tilde{K}$ is the universal cover of K and $C_2(\tilde{K})$ and $C_1(\tilde{K})$ denote the respective chain groups of $\tilde{K}$. Moreover, it follows from the asphericity of knots [7] that there are no non-trivial 2-cycles. Thus the structure of the chain complex is completely determined. Spinning K about $\partial C = S^2$, we obtain a 3-dimensional CW-complex K^* which is a deformation retract of $S^4 - k(S^2)$. The structure of the chain complex of the universal cover $\tilde{K}^*$ of K^* is determined from the previous chain complex. This enables us to calculate $H_2(\tilde{K}^*)$, and hence, by Hurewicz's theorem, $\pi_2(S^4 - k(S^2))$.

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§2. REIDMEISTER HOMOTOPY CHAINS AND (3, 1)-KNOTS

Let $k \subset R^3$ be a polygonal knot in $S^3 = R^3 \cup \infty$, where R^3 is Euclidean 3-space. Let C be a polyhedral 3-cell in S^3 intersecting k in an arc A so that $C \cup k$ is unknotted.

LEMMA. *In $C - k$ there is a 2-dimensional CW-complex K consisting of one vertex v, $n+1$ edges $\xi_0, \xi_1, \ldots, \xi_n$ and n faces $\rho_1, \ldots, \rho_n$ such that*

(a) $\partial C \cap K = v \cup \xi_0$

(b) *there is a deformation retraction of $C - k$ onto K that induces on ∂C a deformation retraction onto $v \cup \xi_0$.*

Proof. Let T be a triangulation of R^3 that has k as a subcomplex and has ∂C as a subcomplex of the dual triangulation $T^{\#}$ where C is a 3-simplex. Adjoin to k any edge of T that lies within C and meets k at one end point only. Continue this process as long as possible (which is only a finite number of times), always adjoining an edge of T within C which meets the previously constructed complex at one end point only. There results a 1-dimensional subcomplex k' that meets ∂C in just two points and has k as a deformation retract. Let K' be the subcomplex of $T^{\#}$ made up of those cells that lie in $C \cup \partial C$ and do not meet k'. Then K' is a complex of dimension ≤ 2, and K' is a deformation retract of $C - k'$, hence of $C - k$. Furthermore $K' \cap \partial C$ is ∂C minus two of its open faces. So a cell-complex K of the required type is obtained by shrinking to a point the two remaining cells plus a suitable maximal tree of K'.

It now follows from this lemma that:

$$G = \pi_1(S^3 - k) \approx \pi_1(C - k) \approx \pi_1(K) = |x_0, x_1 \ldots, x_n : r_1, \ldots, r_m|,$$

where x_j is carried by ξ_j and r_i by ρ_i.

Let $\tilde{K}$ be the universal cover of K. Hence, $\tilde{K}$ is of the same homotopy type as the universal cover of $C - k$. In $\tilde{K}$ we have 0-cells gv, 1-cells $g\xi_0, g\xi_1, \ldots, g\xi_n$, 2-cells $g\rho_1, \ldots, g\rho_n$, where g ranges over G. Then the boundary homomorphisms of the chain complex

$$\ldots \to 0 \to C_2(\tilde{K}) \to C_1(\tilde{K}) \to C_0(\tilde{K})$$

of $\tilde{K}$ are defined by

$$\partial_1 g\xi_j = g\partial_1 \xi_j = g(x_j - 1)v$$

$$\partial_2 g\rho_i = g\partial_2 \rho_i = \textstyle\sum_{j=0}^{n} (\partial r_i/\partial x_j)\xi_j. \qquad \text{See [5, 8, 9, 10].}$$

Remark. $x_j - 1$ and $\partial r_i/\partial x_j$ denote the image of elements of $JF(\mathbf{x})$ in $J\pi_1(S^4 - k(S^2))$.

It now follows from the asphericity of knots [7] *and Hurewicz's theorem that there are no non-trivial 2-cycles.*

§3. SPUN KNOTS

If we now spin K about $\partial C = S^2$, we get a 3-dimensional CW-complex K^* that is a deformation retract of the spun knot $S^4 - k(S^2)$. K^* (which contains K) has a single vertex v; its 1-cells are $\xi_0, \xi_1, \ldots, \xi_n$; its 2-cells are $\rho_1, \ldots \rho_n$ and $\xi_1^*, \ldots \xi_n^*$, where ξ_j^* is swept out by ξ_j during the rotation; and its 3-cells are $\rho_1^*, \ldots, \rho_n^*$, where ρ_i^* is swept out by ρ_i.

In the universal cover $\tilde{K}^*$ we have vertices gv, edges $g\xi_0, g\xi_1, \ldots, g\xi_n$, 2-cells $g\rho_1, \ldots, g\rho_n, g\xi_1^*, \ldots\ g\xi_n^*$, and 3-cells $g\rho_1^*, \ldots, g\rho_n^*$, where g ranges over G. Then the boundary homorphisms of the chain complex

$$\ldots \to 0 \to C_3(\tilde{K}^*) \to C_2(\tilde{K}^*) \to C_1(\tilde{K}^*) \to C_0(\tilde{K}^*)$$

are given by

$$\partial_1 g\xi_j = g\partial_1 \xi_j = g(x_j - 1)v$$

$$\partial_2 g\rho_i = g\partial_2 \rho_i = g \textstyle\sum_{j=0}^{n}(\partial r_i/\partial x_j)\xi_j$$

$$\partial_2 g\xi_j^* = g\partial_2 \xi_j^* = 0 \qquad (j = 1, \ldots, n) \qquad \text{(Fig. 1.)}$$

$$\partial_3 g\rho_i^* = g\partial_3 \rho_i^* = g \textstyle\sum_{j=1}^{n}(\partial r_i/\partial x_j)\xi_j^*. \qquad \text{(Fig. 2.)}$$

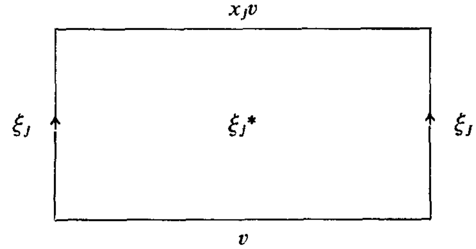

FIG. 1

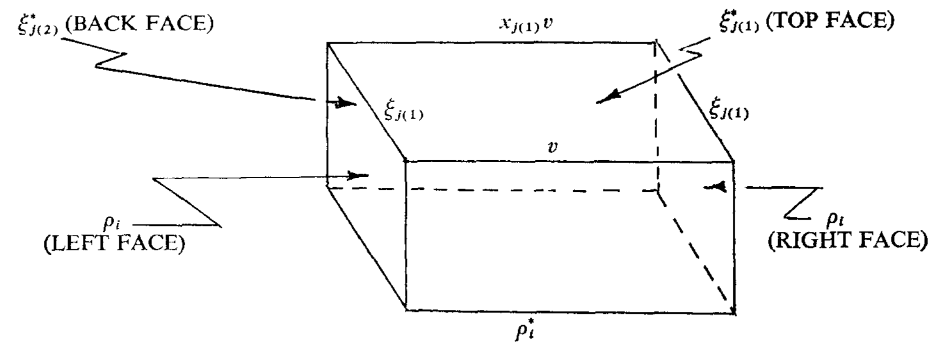

FIG. 2

It follows from the asphericity of knots [7] that the 2-cycles $g\xi_i^*(i = 1, \ldots, n)$ form a basis for the group of 2-cycles of $\tilde{K}^*$. For otherwise there would be a non-trivial 2-cycle in $C_2(\tilde{K})$. Thus as a $J\pi_1$-module:

$$H_2(\tilde{K}^*) = |\xi_1^*, \ldots, \xi_n^* : \textstyle\sum_{j=1}^{n}(\partial r_i/\partial x_j)\xi_j^*(i = 1, \ldots, n)|.$$

By Hurewicz's theorem $\pi_2(S^4 - k(S^2)) \approx \pi_2(K^*) \approx \pi_2(\tilde{K}^*) \approx H_2(\tilde{K}^*)$ as abelian groups. It is now easy to see that the action of $\pi_1(S^4 - k(S^2))$ on $\pi_2(S^4 - k(S^2))$ is the same as that of $\pi_1(S^4 - k(S^2))$ on H_2 $(\tilde{K}^*)$.Hence, $\pi_2(S^4 - k(S^2)) \approx H_2(\tilde{K}^*)$ is a $J\pi_1$-isomophism, and we have proved theorem 2 for one particular presentation of $\pi_1(S^4 - k(S^2))$. If we now make use of the transformations on the presentation of $\pi_2(S^4 - k(S^2))$ as a $J\pi_1$-module induced by Tietze I and Tietze II operations on the presentation of π $(S^4 - k(S^2))$, then the general theorem follows.

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