UNCOUNTABLY MANY MILDLY WILD NON-WILDER ARCS[1]

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In [5] Fox and Harrold defined a *Wilder arc* as a mildly wild L.P.U. (locally peripherally unknotted) arc and then gave a complete classification of such arcs. In this paper, to show that the L.P.U. condition is essential, uncountably many mutually nonequivalent mildly wild non-L.P.U. arcs are constructed.

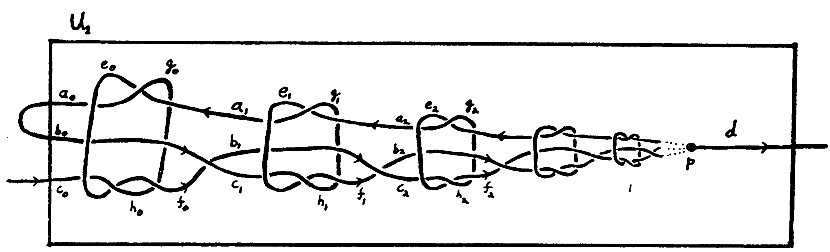

FIGURE 1. A_0

I. **The basic example A_0 (Figure 1).** A regular normed projection of our basic example of a mildly wild non-Wilder arc is shown in Figure 1. (Using the methods of [4], one could easily give a precise description.)

(A) A_0 is not L.P.U. at p. The invariants of [7] will be used to show that the penetration index $P(A_0, p)$ of A_0 at p is equal to 4. (For a definition of the penetration index see [1] and [2].)

The fundamental group $\pi_1(U_1 - A_0)$ of the complement of A_0 in the 3-cell neighborhood U_1 of p is generated by elements a_n, b_n, c_n, d, e_n, f_n, g_n, h_n $(n \geqq 0)$ indicated in the usual way in Figure 1. A set of defining relations is

$$g_n = e_n a_n e_n^{-1}, \qquad a_{n+1} = g_n e_n g_n^{-1}, \qquad h_n = e_n c_n e_n^{-1}, \qquad f_n = h_n e_n h_n^{-1},$$

$$c_{n+1} = e_n b_n e_n^{-1}, \qquad a_{n+1} g_n a_{n+1}^{-1} = c_{n+1} f_n h_n f_n^{-1} c_{n+1}^{-1}, \qquad b_{n+1} = c_{n+1} f_n c_{n+1}^{-1},$$

$$d = a_n^{-1} b_n c_n,$$

where $n \geqq 0$. (The methods of [4] were used to find $\pi_1(U_1 - A_0)$.)

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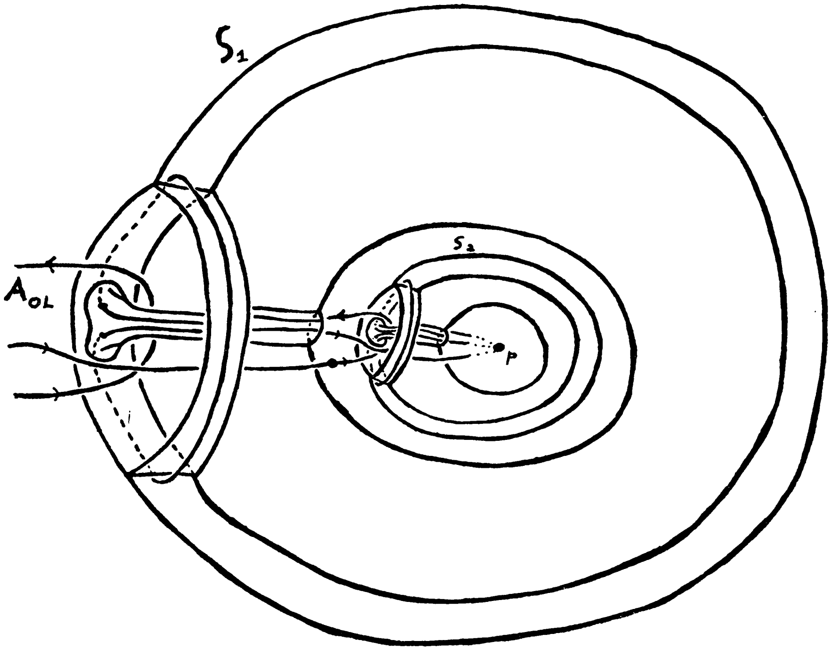

FIGURE 2

Simplifying the above presentation, one finds that $\pi_1(U_1 - A_0)$ is generated by a_n, c_n, e_n ($n \geqq 0$) with the following defining relations

$$e_n a_n e_n a_n^{-1} e_n^{-1} a_{n+1}^{-1} = 1,$$
$$e_n^{-1} a_{n+1}^{-1} c_{n+1} e_n c_n e_n c_n e_n^{-1} c_n^{-1} e_n^{-1} c_{n+1}^{-1} a_{n+1} e_n a_n^{-1} = 1,$$
$$e_{n+1} c_{n+1} e_n c_n e_n c_n^{-1} e_n^{-1} c_{n+1}^{-1} e_{n+1}^{-1} c_{n+2}^{-1} = 1,$$

where $n \geqq 0$. Hence, the corresponding JZ-module is generated by a_n, c_n, e_n ($n \geqq 0$) with the following corresponding defining relations:

$$(t - t^2)a_n + (1 - t + t^2)e_n - a_{n+1} = 0 \qquad (n \geqq 0),$$
$$-t^2 a_n + (t^2 - t^3 + t^4)c_n + (t^3 - t^4)e_n - (1 - t)a_{n+1} + (1 - t)c_{n+1} = 0,$$
$$(t^3 - t^4)c_n + (t^2 - t^3 + t^4)e_n + (t - t^2)c_{n+1} + (1 - t)e_{n+1} - c_{n+2} = 0,$$

where t denotes the generator of the free cyclic group Z and JZ denotes the integral group ring of Z. Thus, the 0th local topology $\Lambda_0(A_0, p)$ of A_0 at p is $\Lambda_0(A_0, p) = (1 - t)^\infty$.

From V.A.7 of [7] the appearance of the factor $1 - t$ in $\Lambda_0(A_0, p)$ implies that $P(A_0, p) > 2$, i.e. that A_0 is not L.P.U. at p. Moreover, since $P(A_0, p)$ must be even, it easily follows that $P(A_0, p) = 4$.

(B) A_0 is mildly wild.

That A_0 is actually mildly wild can be seen by constructing in the obvious fashion an isotopy taking the left half A_{0L} of A_0 onto a straight line segment. A more instructive but unfortunately more complicated method for showing that A_{0L} is tame is to observe that its penetration index [1], [2] at p is equal to 1. This can be seen from Figure 2 where a sequence of 2-spheres converging to p and intersecting A_{0L} at only one point is indicated.

II. **Uncountably many examples.** Let $\Delta_1(t), \Delta_2(t), \Delta_3(t), \cdots$ be an indexing of all mutually distinct (up to associates) prime polynomials in t over the ring J of integers such that:

(i) $\Delta_i(1) = \pm 1$;

(ii) $\Delta_i(t) = t^{2\lambda}\Delta(1/t)$.

By [6], [8] there is a corresponding sequence of distinct knot types $K_1, K_2, K_3, \cdots$ whose Alexander polynomials are respectively $\Delta_1, \Delta_2, \Delta_3, \cdots$.

We may assume that each K_i is a prime knot type. For if K_i is not prime, then by [9], [10] it can be uniquely factored into a product of prime knot types, i.e.

$$K_i = K_{i1} \# K_{i2} \# \cdots \# K_{in},$$

where each K_{ij} is prime. Hence,

$$\Delta_i = \Delta(K_i) = \Delta(K_{i1})\Delta(K_{i2}) \cdots \Delta(K_{in}),$$

where $\Delta(K)$ denotes the Alexander polynomial of K. Since JZ is a unique factorization domain and $\Delta_i(t)$ is prime, there must be a j such that $\Delta(K_{ij}) = \Delta_i(t)$. Hence, K_i may be replaced by the prime K_{ij}.

Let S be the collection of all sequences of 0 or ∞, i.e. $S = \{0, \infty\}^N$, where N denotes the set of natural numbers. *For each s in S let W_s be the Wilder arc in which each K_i appears exactly $s(i)$ times ($i = 1, 2, 3, \cdots$) and no other knot types appear.* (See [5].) Thus, by V.A.7 of [7] the 0th local topology $\Lambda_0(W_s)$ of W_s is

$$\Lambda_0(W_s) = \prod_{i=1}^{\infty} \Delta_i^{s(i)}.$$

For each s in S, let $A_s = A_0 \$_p W_s$, where "$\$_p$" denotes the interior composition of arcs defined in V.C.7 *of* [7]. By V.C.11 of [7],

$$\Lambda_0(A_s) = \Lambda_0(A_0) \cdot \Lambda_0(W_s) = (1 - t)^{\infty} \prod_{i=1}^{\infty} \Delta^{s(i)}.$$

(A) The arcs A_s are all mutually nonequivalent.

With the exponent defined in [7] it is easy to show that all the topologies $\Lambda_0(A_s)$ are distinct. Hence, by the Fundamental Invariance Theorem of [7] the arcs A_s are all mutually nonequivalent.

(B) Not one of the arcs A_s is L.P.U.

As in I.A. of this paper the appearance of $1-t$ in $\Lambda_0(A_s)$ implies that $P(A_s)=4$.

(C) Each of the arcs A_s is mildly wild.

A_s can be shown to be mildly wild in the obvious fashion. One can first apply an isotopy to the left half of A_s to straighten out the A_0 factor and then another in turn to straighten out the W_s factor. Since the right half of A_s is the right half of a Wilder arc, it is also tame.

Hence, $\{A_s: s\in S\}$ is an uncountable collection of mutually nonequivalent mildly wild non-Wilder arcs.

III. **Conjecture.** The following conjecture is based on the examples in this paper as well as many others.

CONJECTURE. If A is a mildly wild arc, then

$$\Lambda_0(A) = (1-t)^\infty \prod_{i=1}^{\infty} \Delta_i(t)^{s(i)},$$

where $\Delta_i(t)$ is a knot polynomial and $s(i)=0$ or ∞.

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