15 May 2023

Knots and Physics

Roger Fenn

School of Mathematical Sciences, University of Sussex

Falmer, Brighton, BN1 9RH, England

Louis H Kauffman

University of Chicago, Illinois 60607, USA

Samuel J Lomonaco Jr.

University of Maryland Baltimore County

Baltimore, MD 21250 USA

My soul is an entangled knot,
Upon a liquid vortex wrought
By Intellect in the Unseen residing.
And thine doth like a convict sit,
With marlinespike untwisting it,
Only to find its knottiness abiding;
Since all the tool for its untying
In four-dimensional space are lying.

J.C.Maxwell (1831-1879)

1.1 Introduction

Knots and physics are related at many levels. Knots and their properties form a dense, visual and complicated affair. See Figure 1 for a depiction of the right handed trefoil knot, the simplest knot. In this paper we tell about knots and topology starting with the theories of Lord Kelvin [**Kelvin 8**] in the nineteenth century and continuing to the present day. Kelvin's work, coming from a deep conviction that knots are fundamental to physics,was instrumental in staring the mathematical theory of knots. After that, the relationships between knots and physics evolved in most remarkable ways. We shall tell an outline of that story.

In the 19-th century, a luminiferous aether permeating all of space was postulated to provide a medium for carrying electromagnetic radiation. Lord Kelvin (Sir William Thompson) further postulated that three dimensional knotted swirls (vortices) in this fluidic aether were atoms of matter. He then tried his hand at defining and modeling vortices in three dimensional fluids. Kelvin's knotted vortices were eventually dismissed along with the notion of the aether as a fluid filling all the regions of the universe. The aether went into the dustbin of history in the same way as phlogiston. The vortex idea has a long gestation and refuses to die. One can find its echoes in many aspects of modern physics. We shall discuss some of these in this article.

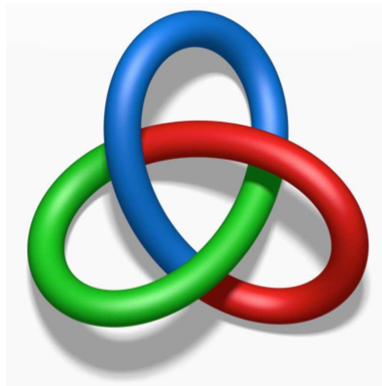

Fig. 1 *A trefoil knot*

One of the first 19th century depiction of knotted vortices is shown in the top half of Figure 2. While earlier work had theorized about knotted fluid vortices, the work of Kleckner and Irvine in 2012 was the first actual production of such vortices under laboratory conditions, in this case in the medium of water.

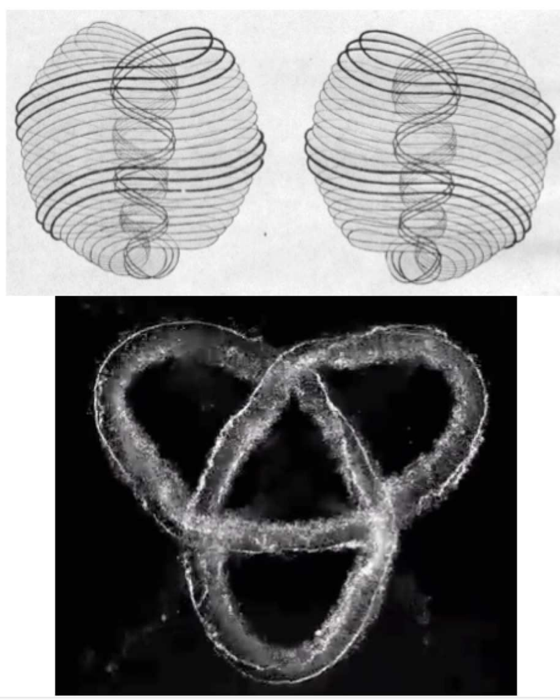

Fig. 2 *Knotted Vortices*

In those early days, Kelvin and Maxwell had their hands on crucially fundamental ideas. The electromagnetic waves were produced by the linking of circulating electricity with circulating magnetism. Each circulation supporting and generating the other to go forth at the speed of light. It quickly became apparent that light itself is an electromagnetic wave, [**University Physics 10**]. See Figure 3.

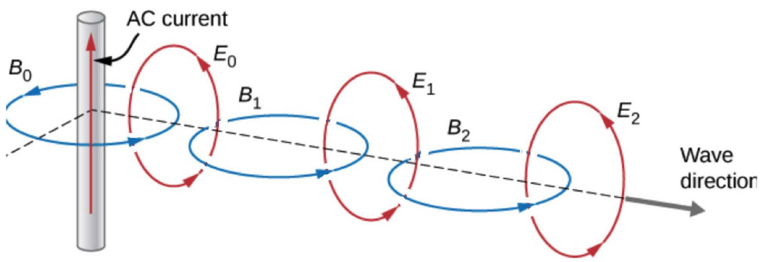

Fig. 3 *The propagation of the electromagnetic wave is based on EM field linking*

Linking between curves is just the beginning of the self-linkedness of the knotting phenomenon.

It is no wonder that Maxwell and Kelvin were convinced that knots held the answers to a better understanding of the physical world. The idea that atoms are knotted vortices in the luminiferous aether is a shot in the dark, illuminated by the electromagnetic theory of light.

Knots have been found to appear in long chain molecules, leading to studies of knotted DNA and knotting in the folding of proteins. In the case of DNA, knot theory is instrumental in understanding the properties of the local geometry of

DNA recombination.

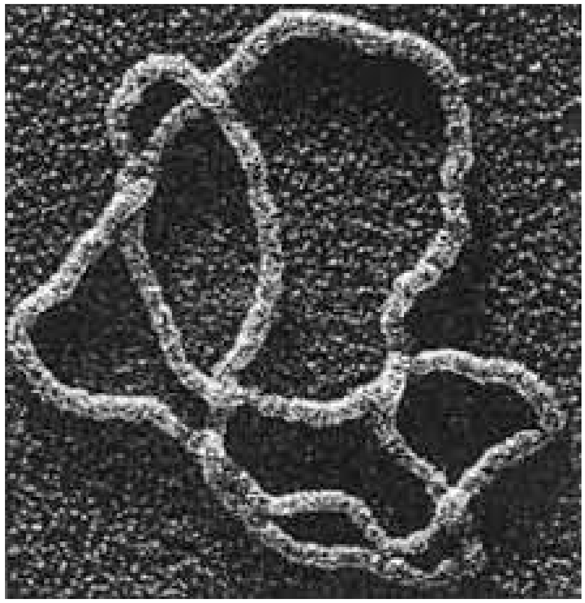

Fig. 4 *A knotted DNA chain*

Cozzarelli, Dungan and Wasserman produced the electron microscope photo in Figure 4.

For many years after Kelvin, knot theory took a mathematical route. Tait and Kirkman and Little made tables of knots and got people started thinking about diagrams and the fundamental topology of those diagrams. Alexander and Briggs and Reidemeister defined diagrammatic knot theories and J. W. Alexander created a polynomial invariant of knots in the 1920's. Knot theory took off and became an integral part of the developing algebraic topology. Methods were developed that could study the topology of the complementary space of the knot – that space where the knot is not. See Figure 5

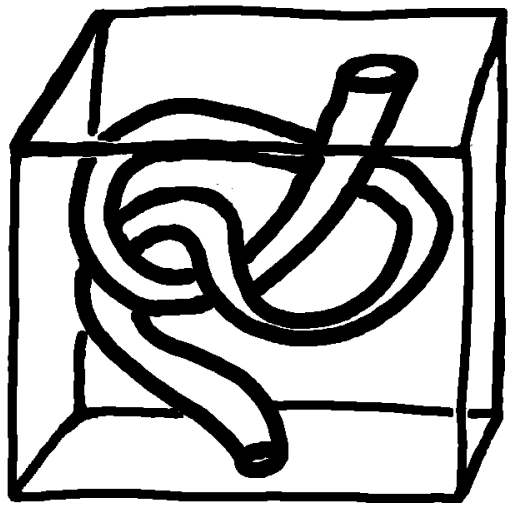

Fig. 5 *The complement of the trefoil knot as the exterior of a knotted tube*

Figure 6 shows how seemingly unrelated topics in physics and mathematics can come together in a light bulb moment. Ideas often tunnel along beneath the surface and then come back again and so the following happened. In the 1960's C.N.Yang studied a "toy" quantum field theory with one dimension of space and one dimension of time, while R Baxter was studying the statistical mechanics of two dimensional systems. They independently discovered that their problems would be illuminated by matrices that satisfied the so-called Yang-Baxter Equation. See Figure 6

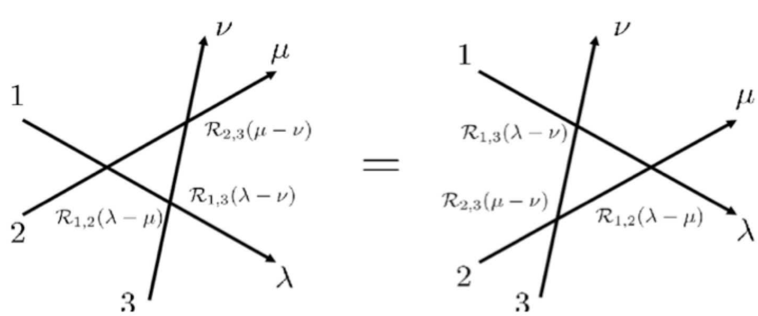

Fig. 6 *The Yang-Baxter Equation*

Today any knot theorist on seeing this diagram, which comes from statistical mechanics would immediately say, "aha, this is an example of the Reidemeister three move", as we shall shortly see.

The Reidemeister moves are how the knot theorists understand and manipulate their objects of study which are *knot diagrams*. If we take the trefoil knot example shown earlier and strip it down to its essentials we get a knot diagram as in Figure 7.

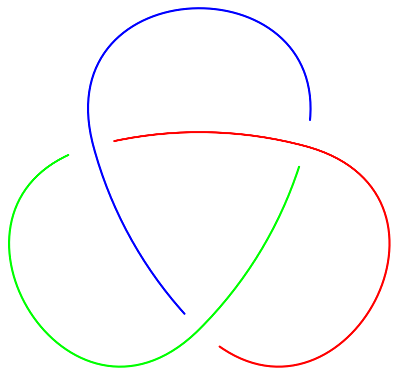

Fig. 7 *The skeleton of a trefoil knot.*

A knot diagram is made up of a number of arcs which end as the knot appears to duck under another arc and start where the knot reappears. The diagram of the knot is a schematic for weaving the knot. It is a line drawing with crossings that indicate where a rope goes over and under itself. We use the same drawing convention that a person does when they make a sketch. (An unbroken arc is closer to the eye of the viewer than a broken arc.) With the diagrams we can record knots and links, and we can regard the diagram as a mathematical notation for the knot or link.

Knot theory is about the *placement problem*, a point of view emphasized by the knot theorist Ralph Fox [**Fox 4**]– how can one understand the embeddings of one topological space A in another topological space B? In the case of knotted rope we take the space A to be an arc (for a length of rope) or a circle (for a closed loop of rope). In itself a circle is very simple, but the myriad ways in which a circle can be embedded in three dimensional space constitutes the mathematical ground of classical knot theory in three dimensions. In that theory, we consider the embeddings up to the equivalence relation of ambient isotopy. Two embeddings K and K' of the circle are said to be ambient isotopic if there is a continuous parametrized family of embeddings, starting with K and ending with K'. This is a model of moving the rope in space to change one knot to the other.

Looking at knot theory in terms of the placement problem, allows mathematicians to consider many different knotting problems such as embeddings of graphs and networks in three dimensional space, embedding surfaces in four dimensional space and embedding high dimensional manifolds in each other. In fact there is a whole raft of theory concerning embedded n dimensional spheres in $n + 2$ dimensional space, the codimension two placement problem.

Returning to the situation in three dimensions, the ambient isotopy or motion of the knot in space is codified by a series of moves shown in Figure 8, that act upon diagrams of knots. These are called Reidemeister moves after Kurt Reidemeister who wrote the first book on knot theory, [**Reidemeister 9**]. The moves were originally discovered by J.C. Maxwell in the middle 1800's. It turns out that Maxwell was genuinely fascinated by knots and he had extensive correspondence about them with other scientists such as Peter Guthrie Tait (a mathematician) and William Thompson (Lord Kelvin). In fact the definition of linking number in classical knot theory is directly related to work by Gauss and Maxwell, see [**Fenn 3**] The completeness of the moves was proved by Alexander and Briggs in the 1920's.

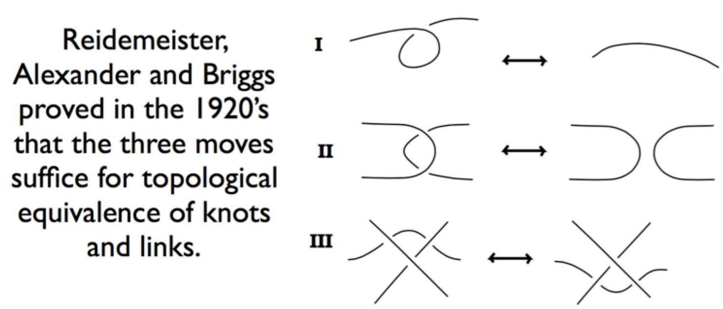

Fig. 8 *The Reidemeister moves*

The connection between Reidemeister move III and the Yang-Baxter equation should now be apparent. From the physical perspective, the Yang-Baxter equation can be interpreted in terms of quantum amplitudes (Yang) and also in terms of partition functions for statistical mechanics (Baxter). A topologist can see the equation as an expression of invariance under the third Reidemeister move, the most complicated of the moves that generate equivalences of knots, and the beginning of the construction of an invariant of knots and links.

The work of Yang and Baxter was appreciated by physicists and particular progress was made by the Russian physicist Ludwig Faddeev and his then-student Nicolai Reshetikhin. They found algebraic ways to understand the Yang Baxter equation. By the early 1980's these methods of Fadeev, Reshetikhin and Takhtajan inspired the mathematician Vladimir Drinfeld to reinvent the theory of Lie algebras in a new form that contained solutions to the Yang-Baxter Equation. A revolution in mathematics and mathematical physics had begun.

This is where our modern story relating knots and physics starts. The path led into a statistical mechanical knot theoretic revolution, spearheaded by Faddeev, Reshetikhin, and Jones. The diagrammatic system of the knots and the Reidemeister moves has an extraordinary reach into mathematics and physics.

Knots are by no means confined to knotted curves in three dimensions. In higher dimensions, a 2-sphere can be knotted in 4-space, a 3-sphere can be knotted in 5-space. One approach to higher dimensional knot theory is to study such higher dimensional knots in terms of hyperplane cross sections. Lomonaco showed that 4-D knot theory could be simplified to 3-D knot theory by judiciously choosing a 3-D cross-section, called the midsection. He went on to show how 5-D knot theory could be reduced to the study of movies of dynamically changing 3-D midsections.

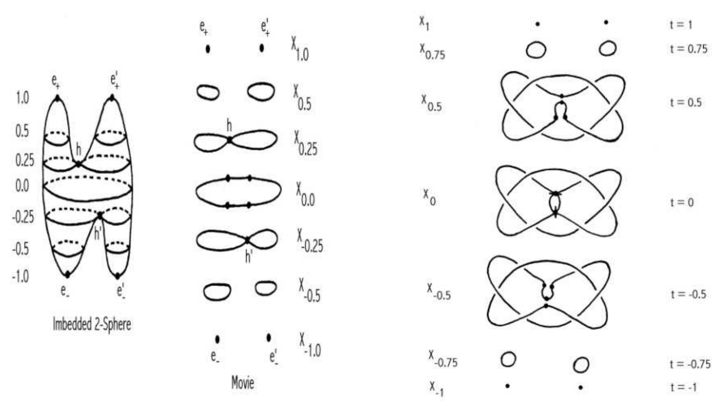

Fig. 9 *3-D cross sections*

On the left of Figure 9 is an example of a movie consisting of three dimensional cross-sections of an unknotted 2-sphere in 4-space. On the right is a movie of 3-D cross-sections of a non-trivially knotted 2-sphere in 4-space, informally called a knotted balloon in 4-space, with midsection shown as the center frame. Fenn and Rolfsen were the first to show that 2-dimensional spheres could link *homotopically* in a 4-dimensional sphere. In a further development Fenn showed that these linked spheres were the singular leaves of a foliation of the 4-dimensional sphere by tori.

Papers by Roger Penrose on spin networks turned out to be a key to knot invariants. Penrose used diagrams and diagrammatic topology, but it was not obvious early on what the spin networks had to do with knots. They were originally invented by Penrose to form a combinatorial model of quantum interactions prior to the emergence of space and time. The q-deformed spin networks (a new variable q is added to the structure) turn out to model augmented knotted networks that represent the topology of three and four dimensional manifolds. The q-deformed basic spin nets give rise to the Kauffman bracket version of the Jones polynomial (that we discuss below) and are a basis for one way to formulate topological quantum computing.

An important knot theorist, John Horton Conway [**Conway 2**], generalized Alexander's method for calculating the important Alexander polynomial [**Alexander 1**]. Figure 10 shows a positive and a negative crossing and a *smoothing* obtained by reconnecting the arcs at the crossing so that the weave disappears. By changes of this sort a knot diagram can be reduced to a collection of disjoint circles. Conway used the formula

$$\nabla_{L_+}(t) - \nabla_{L_-}(t) = z\nabla_{L_0}(t)$$

for the reduction, a mathematical miracle. This is called the Conway skein relation.

Fig. 10 *Two crossings and a smoothing*

In 1979 Kauffman [**Kauffman 6**] found a model for Conway's formula that involved linking numbers and surfaces in space. Then Kauffman went back to Alexander's 1928 paper and discovered that Alexander's determinant for the polynomial could be reformulated as a statistical mechanics type summation, a state sum, the exact analogue of a partition function in statistical mechanics. The polynomial could be seen as a sum over all the ways that a particle could go through the network of the knot diagram without retracing its steps. By summing over all these paths (as in a Feynman path integral) we obtain a polynomial that is a topological invariant of the knot. We obtain the Alexander-Conway polynomial as a state summation.

In 1984, Vaughan Jones [**Jones 5**] found a variant of the Conway skein formula that gave rise to a new invariant now called the Jones polynomial. Jones discovered his invariant by studying the properties of an algebra called the Temperley-Lieb algebra, that is used in statistical mechanics. He rediscovered the Temperley-Lieb algebra from his own deep study of von Neumann algebras, Very closely related to quantum mechanics, Jones construction was generalized by Homflypt. That is an acronym for Hoste, Ocneanu, Millett, Freyd, Lickorish, Yetter, Przytycki and Trawczk. These are mathematicians who heard Jones' early lectures. They found a two variable generalization of the Jones polynomial that is, of course, called the Homflypt polynomial. Jones showed that his new polynomial satisfied a skein relation similar to the Conway skein relation. He proved that

$$t^{-1}V_{L_+}(t) - tV_{L_-}(t) = (\sqrt{t} - 1/\sqrt{t})V_{L_0}(t).$$

For the notation L_+ etc refer to Figure 10

Kauffman searched for a state sum model of Homflypt and of the Jones polynomial. Some months went by and a fortuitous event occurred. Lickorish, Millett and Ho discovered a one variable unoriented skein polynomial. Kauffman found a new two variable unoriented skein polynomial, flew to Italy a couple of days after this discovery, and on the plane found that there was a natural state sum associated with a special case of the new invariant. One can write the equations to describe the state summation as shown below.

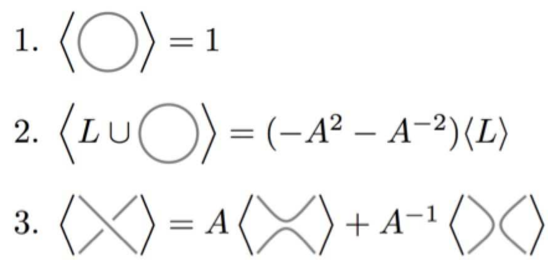

Fig. 11 *Bracket polynomial skein relation*

In this manner the Kauffman bracket was born and with an extra condition to take into account the writhe we get, from the bracket, a state summation for the Jones polynomial. The bracket model for the Jones polynomial shown in Figure 11 comes along with a new interpretation of the Temperley-Lieb algebra that fits directly with the diagrammatics of the knots. It is remarkable that the structure of knots and their diagrams is related to algebras arising in physics.

The discovery of the Kauffman bracket produced a flurry of related results. Jones, Turaev and Reshetikhin showed how to use those solutions to the Yang Baxter Equation to produce partial models for the Homflypt and Kauffman two variable polynomials and how to create many more new invariants by the same method. It became clear that this was a robust connection with statistical mechanics and and with deformed Lie algebras (quantum groups) marking a deep vein in the structure of topology and physics.

But what about Kelvin and his knots? Some modern scientists still think that Kelvin had it right. Jehle has suggested that elementary particles may be knotted magnetic flux. Perhaps an electron is trapped light circulating itself in a knotted form.

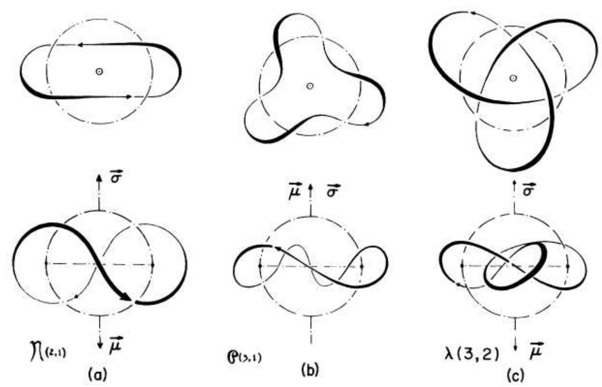

FIG. 3. Forms of quarks in the spinning-top model. These loops represent quarks only if interlinked with other loops as shown in Figs. 4 and 5. The difference of winding numbers about the two dash-dot-dash axes, i.e., $2-1=1(\mathfrak{N})$, $3-1=2(\mathcal{P})$, $3-2=1(\lambda)$, multiplied with the signature of spin with respect to magnetic moment, is proportional to the equivalent electric charge of the respective quarks. Quarks are assumed to be left-handed, antiquarks to be right-handed. Winding numbers have obviously a simple group-theoretical interpretation.

Fig. 12 *Jehle's theory of muons as knots*

Yet another modern day reincarnation of Kelvin's vortex theory are skyrmions, first proposed by Tony Skyrme in the nineteen sixties. Originally used to model properties of nucleons and later finding applications in electromagnetics, solid state physics, string theory, and other areas of physics.

There are many hints that topology is fundamental to the physical structure of the quantum universe. For example, we know that quarks and antiquarks pair up and do not want to come apart, but if you pull them apart they become bound by a narrow string of gluon field. Collisions of protons with protons produce such quark strings and these strings could get knotted and then their ends annihilate to form glueballs, closed loops of gluon field. It has been suggested that the glueballs could be knotted. There is some evidence for this in the work of Niemi and in the work of Kephart and Buiny. These authors suggest that glueballs can indeed be knotted. They even suggest a way to compare the energy levels of the glueballs with a measure of *ropelength* for knots. Ropes with thickness have a length. You can't make a knot in a rope if it is too short. There is a length (for a given diameter) that is just enough for any given knot. We can see this for the simple knots on a rope. Or you can use a computer model to watch a knot being contracted to a minimal ideal form. With computer work we can associate a rope length to each knot. Kephart and Buiny found a strong correlation between the ropelengths of knots and the energy levels of glueballs. Kelvin would have liked this result.

And what about vortices in fluids? Have you seen any knotted vortices lately? Can you blow a knotted smoke ring? Well look at these experiments of William Irvine and Justin Kleckner at the James Franck Institute at the University of Chicago. They made knotted vortices in water. See Figure 13. This is just like Kelvin's dream. And their work opens up new studies in the geometry and topology of fluid vortices. The study of knotted vortices that Kelvin began is just at its beginnings.

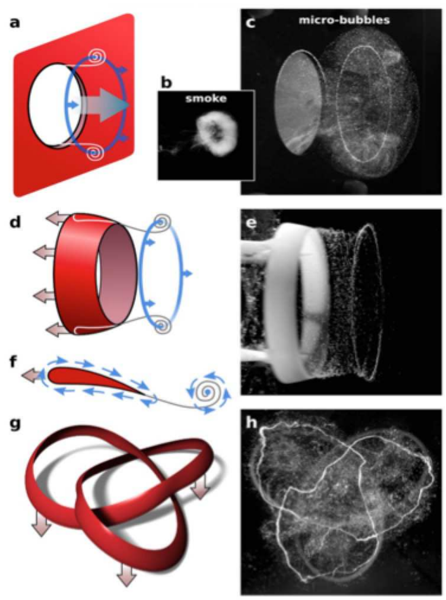

Fig. 13 *Knotted vortices in water - from the work of Kleckner and Irvine*

We are telling you about this shot in the dark of studying knots and linking that began in the 19th century and continues into the present day. Lets go back to the matter of linking. You can see that it makes sense to define a linking number between two curves as the number of times one curve goes around the other.

One can, as Gauss did in the 1800's, measure the way the curves interact in space. Gauss wrote down a integral that measures the linking number directly from how the curves are sitting in space. Then we can use the beautiful notion of the Gauss mapping. Take points on each of the two curves. Then a vector between them defines a direction and hence a point on a two dimensional sphere. The torus of pairs of points on the two curves maps to the sphere and the generic number of points that collapse to a single point, counted correctly, is the linking number! Can we obtain other topological information about knots and links from this prescient idea of Gauss?

Incredibly the answer is yes and it had to wait for 1988 and the insight of Edward Witten [**Witten 11**] who showed how quantum field theory gives topological information about knots. The upshot of his work was a quantum field theoretic interpretation of the Jones polynomial and its relatives and a new way to use quantum field theory to produce topological invariants of three dimensional spaces.The key to Witten's construction is the concept of a gauge connection or gauge potential. The gauge potential is a generalization of an electromagnetic potential wrapped up with a symmetry group in the form of a representation of a Lie algebra. It came about through the initial work of Hermann Weyl and later work of C. N. Yang and Mills to handle nuclear forces in a way that is analogous to electromagnetism but respecting different symmetries. The gauge potential gives a way to track how the internal state of a particle changes as it moves along a path in three dimensional space. If you use the gauge potential to transport the particle around a very tiny loop, you find that the change in its state reflects what is called by mathematicians the curvature of the gauge potential and what is called by physicists the gauge field (electromagnetic field in the case of that theory). Witten understood that knot invariants could be constructed by first measuring the holonomy, the change in the internal space of a particle as the particle is transported around the knot back to its starting point. This is then integrated with the right weights over all possible gauge potentials for a given Lie algebra representation. The result is a knot invariant but it cannot be calculated directly. One can reformulate such concepts until they suggest specific calculations. The Witten Integral can model the Jones polynomial and the Homflypt polynomial and the Kauffman polynomial, all the inventions of Reshetikhin and Turaev and more. And not only that, the Feynman diagram expansion of the Witten integral is written in terms of special space integrals that are direct descendants of the original integrals that Gauss used to find the linking numbers!

We have come full circle and found that what Kelvin, Gauss and Maxwell had wrought, turned into a whole field of topology and physics.

These developments, in the hands of Edward Witten and Michael Atiyah led to what we call Topological Quantum Field Theory, and this area is related to string theory, loop quantum gravity, the development of quantum computers and to the further structure and unification of physics. The topology goes on, and today we study categorifications of the knot invariants. These are new invariants of an algebraic topological nature that yield the knot polynomials as graded Euler characteristics. The categorifications involve new algebra and are related to string theory String theorists work on the knot homologies that arose from categorifiying the Alexander and Jones polynomials using the AdS-CFT correspondence.

And we are not done! The new field of topological quantum computing and quantum topological information has led us to quite different takes on knots and topology. Lomonaco and Kauffman formulate mosaic knots, made from basic tiles as in Figure 14. In that figure the top part of the figure shows a trefoil knot in mosaic form and just below it are illustrated representatives of the basic tiles.

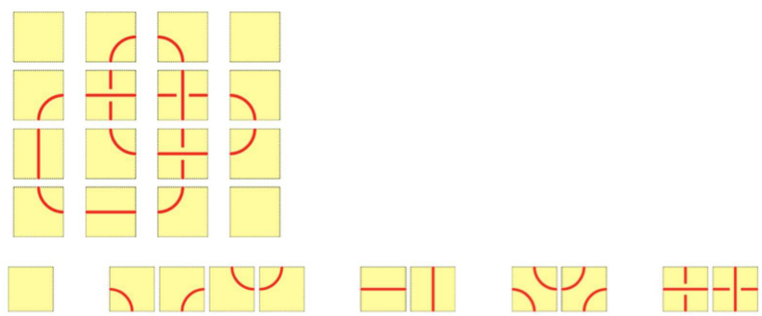

Fig. 14 *Quantum Mosaic Knots*

Kauffman and Lomonaco use the mosaic formulation of knots to represent knots as quantum states in a Hilbert space [**Lomonaco 7**]. Once topological enities can be seen as vectors in a Hilbert space, one can have superpositions of these enities, quantum evolutions and measurements. Topology intertwines with quantum theory in a new way. The basic tiles are regarded as basis elements for a Hilbert space and the mosaic diagram for a knot is seen as a tensor product of the tiles into which it is composed. In this way, the mosaic knot diagram is a vector in a Hilbert space that is a tensor product of copies of the basic tile space. Regarding the knot as a vector in a Hilbert space, it can be seen as a quantum state. Analogs of the Reidemeister moves are represented by unitary transformations of this tensor space. Knot theory is projected into the physical context of quantum information and quantum computing. There is much to say about this level of development, but it is time to stop. Note that we have come round a spiral from knots in the luminiferous aether to knots as states in quantum Hilbert space.

This paper has been a survey of how the topological and geometrical properties of knots and links in three dimensional space are related to physics. The relationships occur at many levels, from the direct phenomena of knotting in rope, materials, vortices and more. Historically knots and physics were linked by Lord Kelvin (Sir William Thomson) in the nineteenth century with his theory of atoms as knotted vortices in the luminiferous aether. This idea has not gone away. In a sense, this paper has been a modern review of the present state of Kelvin's ideas with the evolution of the study of actual vortices in fluids, the use of generalizations of the linking numbers of Gauss and Maxwell in the quantum field theory interpretations of the Jones polynomial by Edward Witten and much more. We ask the reader to ponder the question, just how fundamental are knots and their topology for the science of physics?

References

[1] J.W. Alexander, Topological invariants of knots and links. *Trans. Amer. Math. Soc.* **30**, 275–306 (1928).

[2] J.H. Conway, An enumeration of knots and links and some of their algebraic properties, in *Computational Problems in Abstract Algebra*, Pergammon Press, N.Y.,1970, pp. 329-358.

[3] R. Fenn, Techniques of Geometric Topology Lecture Note Series 57 LMS CUP

[4] R. H. Fox, A quick trip through knot theory, in "Topology of Three-Manifolds", edited by M. K. Fort, Proceedings of the University of Georgia Insititute (1961), Prentice Hall Pub. 1962, pp. 120-167.

[5] V.F.R. Jones, A polynomial invariant of links via von Neumann algebras, *Bull. Amer. Math. Soc.*, 1985, No. 129, pp. 103-112.

[6] L.H. Kauffman, “Knots and Physics”, World Scientific, Singapore/New Jersey/London/Hong Kong, 1991, 1994, 2001, 2012.

[7] S. Lomonaco, Jr. (editor), ”Quantum Information Science and its Contributions to Mathematics”, Proceedings of Symposia in Applied Mathematics, Vol. 68, American Mathematical Society Pub. (2010).

[8] [Kelvin] Thomson, W.T., (Lord Kelvin), On vortex atoms, Philosophical Magazine, 34 July 1867, p15-24, Mathematical and Physical Papers, Vol. 4, Cambridge (1910).

[9] K. Reidemeister, *Knotentheorie*, Chelsea Pub. Co., New York, 1948, Copyright 1932, Julius Springer, Berlin.

[10] University Physics Volume 2. OpenStax CNX. Jul 16, 2019
http://cnx.org/contents/7a0f9770-1c44-4acd-9920- 1cd9a99f2a1e@16.7.

[11] Edward Witten, Quantum field Theory and the Jones Polynomial, *Commun. Math. Phys.*,vol. 121, 1989, pp. 351-399.