THE THIRD HOMOTOPY GROUP OF SOME HIGHER DIMENSIONAL KNOTS

S. J. Lomonaco, Jr.

0. *Introduction*

In 1962 Fox [1] posed the problem of computing the second homotopy group of the complement $S^4 - k(S^2)$ of a (4, 2)-knot as a $Z\pi_1$-module. Although Epstein [3] had previously shown that π_2 as an abelian group (without $Z\pi_1$-action) was algebraically uninteresting, Fox pointed out that this might not be the case when the action of π_1 on π_2 is considered. Since then some progress has been made. In [6, 7, 8] a presentation of the second homotopy group of an arbitrary spun knot [5] was calculated as a $Z\pi_1$-module and found to be algebraically non-trivial. In particular,

THEOREM 0. *If* $k(S^2) \subset S^4$ *is a* 2-*sphere formed by spinning an arc* α *about the standard* 2-*sphere* S^2 *and* $(x_1, \cdots, x_n : r_1, \cdots, r_m)$ *is a presentation of* $\pi_1(S^4 - k(S^2))$, *then*

$$\left(X_1, \cdots, X_n : \sum_i (\partial r_j / \partial x_i) X_i = 0 \ (0 \leq j \leq m)\right)$$

is a presentation of $\pi_2(S^4 - k(S^2))$ *as a* $Z\pi_1$-*module, where* $r_0 = r_0(x_1, \cdots, x_n)$ *is the image of the generator of* $\pi_1(S^2 - \alpha)$ *under the inclusion map and the symbols* $\partial r_j / \partial x_i$ *denote the images of Fox's derivatives* [9] *in* $\pi_1(S^4 - k(S^2))$.

Little appears to be known about the higher dimensional homotopy groups. In this paper a procedure is given for computing a presentation of π_3 of a spun knot as a $Z\pi_1$-module. Specifically, THEOREM 1. *Let* $(S^4, k(S^2))$ *be defined as in Theorem 0 above. Then* $\pi_3(S^4 - k(S^2))$ *is isomorphic as a* $Z\pi_1$*-module to* $\Gamma(\pi_2(S^4 - k(S^2))$, *where* Γ *denotes a functor defined by J. H. C. Whitehead* [10, 11] *and later generalized by Eilenberg and MacLane* [12, 13]. *Hence,* π_3 *as a* $Z\pi_1$*-module is determined by* π_1 *and* π_2.

COROLLARY 2. *If* $\pi_2 \neq 0$, *then* π_3 *of a spun knot as a group (i.e., without* $Z\pi_1$*-structure) is free abelian of infinite rank. Otherwise,* $\pi_3 = 0$.

THEOREM 3. *Let* $k(S^2) \subset S^4$ *be a 2-sphere formed by spinning an arc* a *about the standard 2-sphere* S^2 *and* $(x_1, \cdots, x_n : r_1, \cdots, r_m)$ *be a presentation of* $\pi_1(S^4 - k(S^2))$. *Let* $r_0 = r_0(x_1, \cdots, x_n)$ *be the image of the generator of* $\pi_1(S^2 - k(S^2))$ *under the inclusion map and* X_i *and* $\partial r_i/\partial x_j$ *be as in Theorem 0. Then as a* $Z\pi_1$*-module,* $\pi_3(S^4 - k(S^2))$ *is generated by the symbols*

$$\gamma(X_i),\ [X_i, gX_j] \qquad (1 \le i, j \le n;\ g \in \pi_1)$$

subject to the relations

$$\begin{aligned}
&2\gamma(X_i) = [X_i, X_i] && 1 \le i, j \le n \\
&\gamma\left(\sum_{j=1}^{n} (\partial r_k/\partial x_j) X_j\right) = 0 && 0 \le k \le m \\
&\left[X_i, g \sum_j (\partial r_k/\partial x_i) X_j\right] = 0 && g \in \pi_1 \\
&[X_i, gX_j] = g[X_j, g^{-1}X_i] &&
\end{aligned}$$

where $[X_i, gX_i]$ *is the Whitehead product of* X_i *and* gX_j *and* $\gamma(X_i)$ *is represented by the composition of the Hopf map* $S^3 \to S^2$ *with a representative of* X_i.

Applications of the above theorem to specific examples can be found in the last section of this paper.

I would like to thank Richard Goldstein for his helpful comments during the preparation of this paper and also Peter Kahn for suggesting the above more general formulation of Theorem 0.

REMARK. The methods of this paper may easily be extended to p-spun knots.

I. Definition of a Spun Knot

Let S^2 be a standard 2-sphere in the 3-sphere S^3 and let α be a polyhedral arc with endpoints lying on S^2 and with interior lying entirely within one of the two components of $S^3 - S^2$. (See Figure 1.)

If α is spun about S^2 holding S^2 fixed, a knotted 2-sphere $k(S^2)$ in S^4 is generated [5]. If one would like to think of the spinning as taking place in time, then at time 0, the arc α would appear on the right of the 2-sphere as indicated in the figure. It would then immediately vanish into another 3-dimensional hyperplane and after rotating through 180° suddenly reappear inside S^2 as indicated by the dotted arc on the left of Figure 1. Again it would disappear into another 3-dimensional hyperplane and rotate through the remaining 180° until it suddenly reappeared on the right closing up the knotted 2-sphere $k(S^2)$.

II. $\pi_3 = \Gamma(\pi_2)$

The complement $X = S^4 - k(S^2)$ of an arbitrary spun knot $(S^4, k(S^2))$ will not be examined in more detail. Let $X_0 = S^3 - k(S^2)$ be the 3-dimensional cross-section shown in Figure 1, and X_+ and X_- denote the closures of the two components of $X - X_0$. Let $p : \tilde{X} \to X$ be the universal covering of X and $\tilde{X}_i = p^{-1}(X_i)$ for $i = +,\ 0,$ and $-$.

Since $\pi_1(X_i) \to \pi_1(X)$ are all onto, it follows from the homotopy sequence of the fibration

$$\pi_1(X) \to \tilde{X}_i \to X_i$$

that $\tilde{X}_i$ is connected and

$$1 \to \pi_1(\tilde{X}_i) \to \pi_1(X_i) \to \pi_1(X) \to 1$$

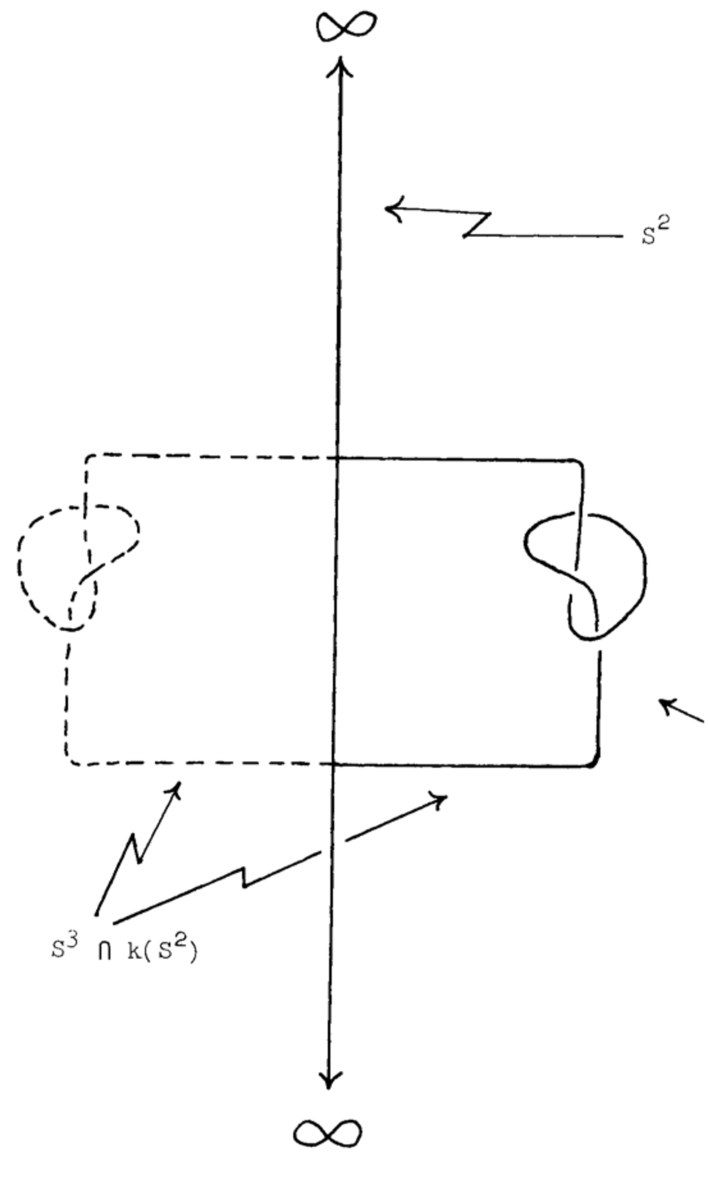

Figure 1. Spun 2-sphere

is exact for $i = +$, 0, and $-$. Moreover, since $\pi_1(X_\pm) \to \pi_1(X)$ is an isomorphism onto [5], it follows that $\tilde{X}_\pm$ are simply connected, and hence are the universal covers of $X_\pm$. Thus,

LEMMA. *The lift $\tilde{X}_0$ of X_0 to the universal cover of X is connected and $\pi_1(\tilde{X}_0)$ is the kernel of $\pi_1(X_0) \to \pi_1(X)$. Moreover, the lifts $\tilde{X}_\pm$ of $X_\pm$ are the universal covers of $X_\pm$.*

Since $\tilde{X}_+$ and $\tilde{X}_-$ both collapse to the right half of $\tilde{X}_0$ via a deformation arising from the spinning, Hurewicz's theorem coupled with the asphericity of knots [4] yields that $H_n(\tilde{X}_\pm) = 0$ for $n \geq 1$. Hence, from the Mayer-Vietoris sequence for the triad $(\tilde{X}; \tilde{X}_+, \tilde{X}_-)$, we have $H_n(\tilde{X}) \simeq H_{n-1}(\tilde{X}_0)$ for $n \geq 2$. Thus,

LEMMA.
$$H_2(\tilde{X}) \simeq H_1(\tilde{X}_0) \quad \textit{and}$$
$$H_n(\tilde{X}) = 0 \quad \textit{for} \quad n > 2 \ .$$

Proof. Since $\tilde{X}$ collapses to a 3-dimensional CW-complex [8], the last part of this lemma is obviously true for $n > 3$. $H_3(\tilde{X}) \simeq H_2(\tilde{X}_0)$ can be shown to be equal to zero by an analysis of the following decomposition of $\tilde{X}_0$.

Let $X_0^\pm$ denote the closure of the two components of $X_0 - S^2$. Then $X_0 = X_0^+ \cup X_0^-$ and $X_{00} = X_0^+ \cap X_0^-$ is S^2 minus the two endpoints of α. Hence, X_{00} is a homotopy 1-sphere and $\pi_1(X_{00})$ is infinite cyclic. Since $\pi_1(X_0^\pm) \to \pi_1(X)$ is an isomorphism onto [5], it follows from the homotopy sequence of the fibration

$$\pi_1(X) \to \tilde{X}_0^\pm \to X_0^\pm$$

that $\tilde{X}_0^\pm$ are simply connected. Applying the asphericity of knots [4], we have that $H_2(\tilde{X}_0^\pm) = 0$. After inspecting the Mayer-Vietoris sequence for the triad $(\tilde{X}_0; \tilde{X}_0^+, \tilde{X}_0^-)$, we have

$$H_2(\tilde{X}_0) \simeq H_1(\tilde{X}_{00}) \ .$$

Since the image of a generator of $\pi_1(X_{00})$ in $\pi_1(X)$ has a linking number of ± 1 with respect to $k(S^2)$, $\pi_1(X_{00}) \to \pi_1(X)$ is a monomorphism. Thus, from the homotopy sequence of the fibration

$$\pi_1(X) \to \tilde{X}_{00} \to X_{00} \ ,$$

we have that $\tilde{X}_{00}$ is simply connected. Hence, $H_3(\tilde{X}) \simeq H_2(\tilde{X}_0) \simeq H_1(\tilde{X}_{00}$ $= 0$.

With the above lemma and J. H. C. Whitehead's Certain Exact Sequence [10, 11], we have

$$\pi_n(X) \simeq \Gamma_n(X) \qquad \text{for } n \geq 3 \ .$$

Hence, $\Gamma_3(X) = \Gamma(\pi_2(X))$, where Γ is an algebraic functor defined by J. H. C. Whitehead [10, 11] and later generalized by Eilenberg and MacLane [12, 13]. This formula gives an effective procedure for computing $\pi_3(X)$. In summary, we have

THEOREM 1. $\pi_3(S^4 - k(S^2)) \simeq \Gamma(\pi_2(X))$. *Hence, the third homotopy group of a spun knot as a* $Z\pi_1$*-module is determined by the first and second homotopy groups. As an abelian group, it is determined solely by* π_2.

From [3], $\pi_2(X)$, if non-zero, is free abelian of infinite rank. Since Γ never decreases the rank of a free abelian group, we have

COROLLARY 2. *The third homotopy group of a spun knot as a group is free abelian of infinite rank if the second homotopy group is non-zero. Otherwise, it is zero.*

III. Whitehead's Functor

A more detailed understanding of J. H. C. Whitehead's functor Γ [10, 11] is needed to compute a presentation of $\pi_3(X)$. Very briefly, Γ is defined as follows. (For more details see [10, 11].)

Let A be an additive abelian group. Then $\Gamma(A)$ is an additive abelian group generated by the symbols

$$\{\gamma(a)\}_{a \in A}$$

subject to the relations

$$\gamma(-a) = \gamma(a) \tag{1}$$

$$\begin{aligned} \gamma(a+b+c) - \gamma(b+c) - \gamma(c+a) - \gamma(a+b) \\ + \gamma(a) + \gamma(b) + \gamma(c) = 0 \ . \end{aligned} \tag{2}$$

Define $[a, b]$ by

$$\gamma(a+b) = \gamma(a) + \gamma(b) + [a, b] \ .$$

Then, $[a, b]$ is a measure of how close γ is to a homomorphism.

The following relations are consequences of (1) and (2).

$$\gamma(0) = 0$$

$$2\gamma(a) = [a, a]$$

$$[a, b+c] = [a, b] + [a, c]$$

$$[a, b] = [b, a]$$

$$\gamma\left(\sum_i a_i\right) = \sum_i \gamma(a_i) + \sum_{i<j} [a_i, a_j]$$

$$\gamma(na) = n^2 \gamma(a) \ .$$

A proof of the following theorem can be found in [10, 11].

THEOREM. *If* A *is an additive abelian group with generators* a_i *and relations* b_j, *then* $\Gamma(A)$ *is an additive abelian group with generators*

$$\{\gamma(a_i)\} \cup \{[a_i, a_j]\}_{i<j}$$

and relations

$$\{\gamma(b_j) \quad 0\} \cup \{[a_i, b_j] = 0\} \ .$$

Finally, if A admits a group of operators W, then so does $\Gamma(A)$, according to the rule

$$w\gamma(a) = \gamma(wa)$$

for $w \in W$ and $a \in A$.

IV. Computation of $\pi_3(S^4 - k(S^2))$

From Section II, $\pi_3(X) \simeq \Gamma(\pi_2(X))$, and from Section III, $\pi_3(X)$ is generated by

$$\{\gamma(\xi)\}_{\xi \in \pi_2(X)} \quad \text{and} \quad \{[\xi, \xi']\}_{\xi, \xi' \in \pi_2(X)} .$$

In [10, 11] J. H. C. Whitehead demonstrates that $[\xi, \xi']$ is the Whitehead product of ξ and ξ' and that $\gamma(\xi)$ is represented by the composition of the Hopf map $S^3 \to S^2$ with a representative of ξ. Hence, we have

THEOREM 3. *Let* $k(S^2) \subset S^4$ *be a 2-sphere formed by spinning an arc* α *about the standard 2-sphere* S^2 *and* $(x_1, \cdots, x_n : r_1, \cdots, r_m)$ *a presentation of* $\pi_1(S^4 - k(S^2))$. *Let* $r_0 = r_0(x_1, \cdots, x_n)$ *be the image of the generator of* $\pi_1(S^2 - k(S^2))$ *under the inclusion map and* X_i *and* $\partial r_i/\partial x_j$ *be as in Theorem 0. Then as a* $Z\pi_1$*-module,* $\pi_3(S^4 - k(S^2))$ *is generated by the symbols*

$$\gamma(X_i), \ [X_i, gX_i] \qquad (1 \leq i, j \leq n;\ g \in \pi_1)$$

subject to the relations

$$\begin{aligned} &2\gamma(X_i) = [X_i, X_i] && 1 \leq i,\ j \leq n \\ &\gamma\left(\sum_{j=1}^{n} (\partial r_k/\partial x_j) X_j\right) = 0 && 0 \leq k \leq m \\ &\left[X_i, g \sum_j (\partial r_k/\partial x_i) X_j\right] = 0 && g \in \pi_1 \\ &[X_i, gX_j] = g[X_i, g^{-1}X_j] , && \end{aligned}$$

where $[X_i, gX_i]$ *is the Whitehead product of* X_i *and* gX_j *and* $\gamma(X_i)$ *is represented by the composition of the Hopf map* $S^3 \to S^2$ *with a representative of* X_i.

V. Examples

EXAMPLE 1. If the trefoil is spun about S^2, then

$$\pi_1(S^4 - k(S^2)) = |a, b : baba^{-1}b^{-1}a^{-1}|$$

$$\pi_2(S^4 - k(S^2)) = |B : (1-a+ba)B = 0 \quad |$$

$$\pi_3(S^4 - k(S^2)) = \left| \gamma(B), [B,gB], (g \in \pi_1) : \begin{array}{r} 2\gamma(B) = [B, B] \\ (1-a+ba)\gamma(B) = -[B, baB] \\ [B,gB] - [B,gaB] + [B,gbaB] = 0 \\ [B,gB] = g[B, g^{-1}B] \end{array} \right|$$

where $[B, gB]$ is the Whitehead product of B and gB and $\gamma(B) =$ (Hopf map) o B. (See Figure 2.)

EXAMPLE 2. If the square knot is spun about S^2, then

$$\pi_1(S^4 - k(S^2)) = |a,b,c : baba^{-1}b^{-1}a^{-1}, caca^{-1}c^{-1}a^{-1}|$$

$$\pi_2(S^4 - k(S^2)) = |B, C : (1-a+ba)B = 0 = (1-a+ca)C|$$

$$\pi_3(S^4 - k(S^2)) = \left| \begin{array}{l} \gamma(B) \\ \gamma(C) \\ [B,gB] \\ [C,gC] \\ [B,gC] \\ [C,gB] \end{array} : \begin{array}{l} 2\gamma(B) = [B,B],\ 2\gamma(C) = [C,C] \\ (1-a+ba)\gamma(B) = -[B,baB] \\ (1-a+ca)\gamma(C) = -[C,caC] \\ [B,gB] - [B,gaB] + [B,gbaB] = 0 \\ [C,gB] - [C,gaB] + [C,gbaB] = 0 \\ [B,gC] - [B,gaC] + [B,gcaC] = 0 \\ [C,gC] - [C,gaC] + [C,gcaC] = 0 \\ [B,gB] = g[B,g^{-1}B],\ [C,gC] = g[C,g^{-1}C] \\ [B,gC] = g[C,g^{-1}B] \end{array} \right|$$

where g ranges over π_1.

COMMUNICATIONS RESEARCH DIVISION
INSTITUTE FOR DEFENSE ANALYSES
PRINCETON, NEW JERSEY
AND
STATE UNIVERSITY OF NEW YORK

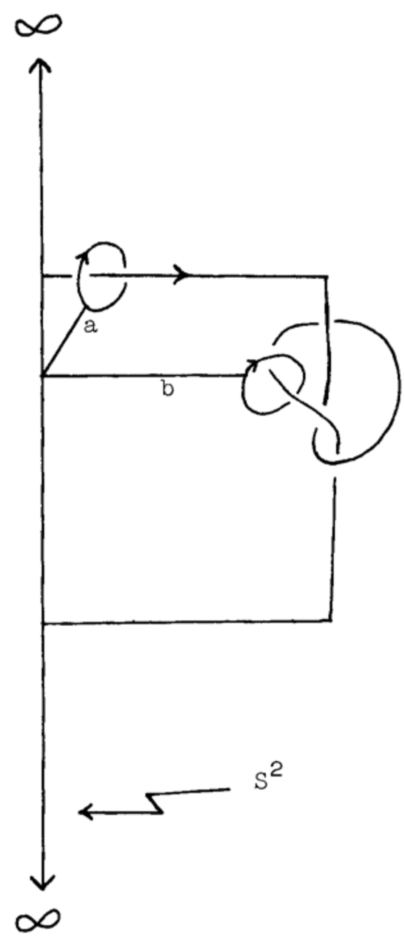

Figure 2. Spun Trefoil

REFERENCES

[1] Fox, R. H., *Some problems in knot theory*, in Topology of 3-Manifolds Symposium, Prentice-Hall, Englewood Cliffs, N. J. 1962

[2] Crowell, R. H., and Fox, R. H., Knot Theory, Ginn, 1963.

[3] Epstein, D. B. A., *Linking spheres*, Proc. Camb. Phil. Soc. math. phys. sci. 56 (1960), 215-219.

4] Papakyriakopoulos, C. D., *On Dehn's lemma and the asphericity of knots*, Ann. of Math. 66 (1957), 1-26.

5] Artin, Emil, *Zur isotopie zweidimensional Flachen im* R^4, Hamburg Abh. 4 (1925), 174-177.

6] Andrews, J. J., and Lomonaco, S. J., *The second homotopy group of 2-spheres in 4-space*, A.M.S. Bull., 75 (1969), 169-171.

7] ______________________, *The second homotopy group of 2-spheres in 4-space*, Ann. of Math. 90 (1969), 199-204.

8] Lomonaco, S. J., Jr., *The second homotopy group of a spun knot*, Topology, 8 (1969), 95-98.

9] Fox, R. H., *Free differential calculus I. Derivations in the free group ring*, Ann. of Math. 57 (1953), 547-560.

10] Whitehead, J. H. C., *The secondary boundary operator*, Proc. Nat. Acad. Sci. 36 (1950), 55-60.

11] ______________, A certain exact sequence, Ann. of Math. 52 (1950), 51-110.

12] Eilenberg, Samuel and Saunders MacLane, *On the Groups* $H(\pi, n)$, I, Ann. of Math. 58 (1953), 55-106; II, Ann. of Math. 70 (1954), 49-139; III, Ann. of Math. 60 (1954), 513-557.

13] MacLane, Saunders, Homology, Academic Press, 1963.

14] Andrews, J. J., and Curtis, M. L., *Knotted 2-spheres in the 4-sphere*, Ann. of Math. 70 (1959), 565-571.

15] Hu,Sze-tsen, Homotopy Theory, Academic Press, 1959.

16] Cappell, Sylvain, *Superspinning and knot complements*, in Topology of Manifolds Symposium, Markham Publishing Co., Chicago. 1970.