Quantum Braids & Mosaics

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This work
is in collaboration with
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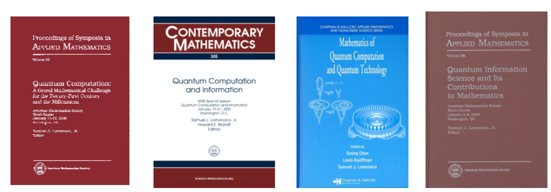
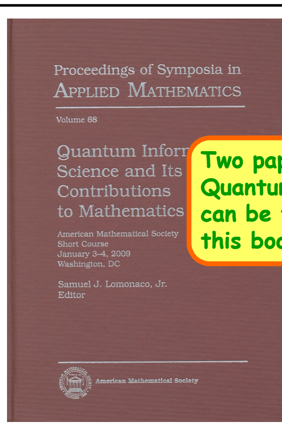




Two papers on Quantum Knots can be found in this book.

This talk was motivated by:

Kitaev, Alexei Yu, Fault-tolerant quantum computation by anyons, arxiv.org/abs/quant-ph/9707021

Wilczek, F., Fractional statistics and anyon superconductivity, World Scientific Press, (1990).

Rasetti, Mario, and Tullio Regge, Vortices in He II, current algebras and quantum knots, Physica 80 A, North-Holland, (1975), 217-2333.

This talk is based on the paper:

Lomonaco and Kauffman, Quantum Braids and Other Mathematical Structures: The General Quantization Procedure, This SPIE Proceedings, (2011).

The above paper distills the ideas found in the following two papers into a general quantization procedure.

Lomonaco and Kauffman, Quantum Knots and Mosaics, Journal of Quantum Information Processing, vol. 7, Nos. 2-3, (2008), 85-115.

Lomonaco and Kauffman, Quantum Knots and Lattices, AMS PSAPM/68, (2010), 209-276

All the above papers can be found on the ArKiv and on the website:

www.csee.umbc.edu/~lomonaco

PowerPoint slides can be found at:
www.csee.umbc.edu/~lomonaco/Lectures.html

This general mathematical procedure can be used to quantize:

- Knots, Graphs, & Braids
- Groups
- Categories
- Algebraic Varieties
- Topological & Differential Manifolds
- And more

- Each particular application of this general procedure creates a blueprint for a physically implementable quantum system.
- These quantum systems are physically implementable in the same sense as Shor's quantum factoring algorithm is physically implementable

Outline of General Quantization Procedure

Step 1. Mathematical construction of a Symbolic Motif System S

Step 2. Mathematical construction of a Quantum Motif System Q based on S

Outline

Mathematical Structure = Formal Rewriting System

⇩

Formal Rewriting System = Group Representation

⇩

Quantum Mechanics = Group Representation Theory

Thinking Outside the Box

Quantum Mechanics

is a tool for exploring

Mathematical Structure
of your choice

We will illustrate the general quantization procedure by showing how it can be used to quantize braids.

Please keep in mind that the same general quantization procedure can be applied to many other mathematical structures.

Quantum Braids

Braiding Naturally Arise in the Quantum World as Dynamical Processes

Examples of dynamical knots and braids naturally occur in quantum physics as Quantum Vortices:

- In supercooled helium II
- In the Bose-Einstein Condensate
- In the Electron fluid found within the fractional quantum Hall effect

Reason for current intense interest:
Topology Is a Natural Obstruction to Decoherence

- Our objective is to do mathematics in such a way that it is intimately related to quantum physics
- Our ultimate objective is to create and to investigate mathematical objects that can be physically implemented in a quantum physics lab.

Objectives

- We seek to create a quantum system that simulates braided physical pieces of rope.
- We seek to define a quantum braid in such a way as to represent the state of braided pieces of rope, i.e., the particular spatial configuration.
- We also seek to model the ways of moving the braid around (without cutting the rope, and without letting it pass through itself.)

Rules of the Game

Find a mathematical definition of a quantum braid that is

- Physically meaningful, i.e., physically implementable, and
- Simple enough to be workable and useable.

Aspirations

We would hope that this definition will be useful in modeling and predicting the behavior of vortices that actually occur in quantum physics such as

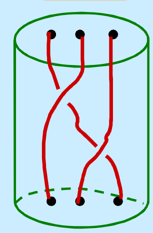
- In supercooled helium II
- In the Bose-Einstein Condensate
- In the Electron fluid found within the fractional quantum Hall effect

What Is the Braid Group B_n ???

Skip braid gp def

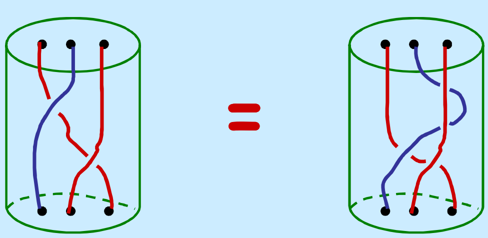
A Braid

Hat Box



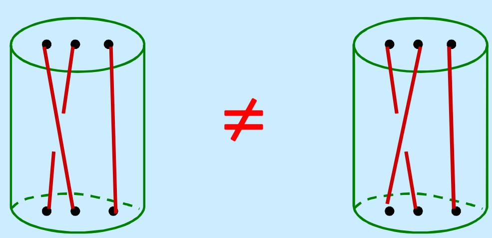
3 Strand braid β

Two Equal Braids



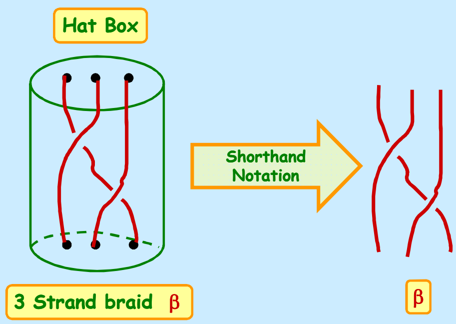
$$\beta_1 = \beta_2$$

Two Unequal Braids

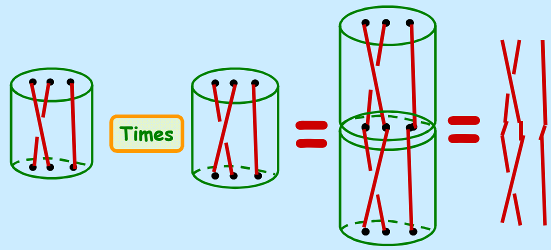


$$\beta_1 \neq \beta_2$$

Shorthand Notation

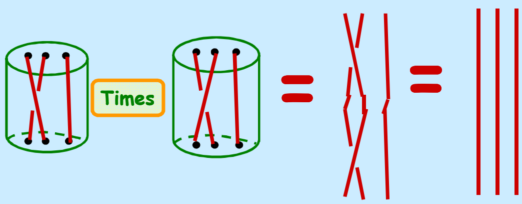


Product of Braids



$$\beta_1 \bullet \beta_2 = \beta_3$$

Inverse of of a Braid



$$\beta \bullet \beta^{-1} = 1$$

To construct the inverse of a braid, take the mirror image of each crossing, and then reverse the order of the crossings.

The n-Stranded Braid Group B_n

Theorem (Emil Artin). Under braid multiplication, the n-stranded braids form a group B_n, call the n-stranded braid group

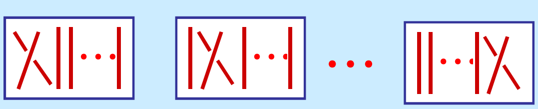
There is a natural monomorphism

$$B_n \to B_{n+1}$$

$$\beta \mapsto \beta' = (\beta \mid\ )$$

Generators of the Braid Group B_n

The braid group B_n is generated by



b_1 b_2 … b_{n-1}

Relations Among the Generators of B_n

Reidemeister 3 Move

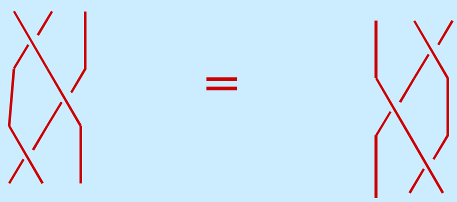
$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \text{ , } 1 \le i < n$$

$$b_i b_j = b_j b_i \text{ , for } |i-j| \ge 2$$

Planar Isotopy Move

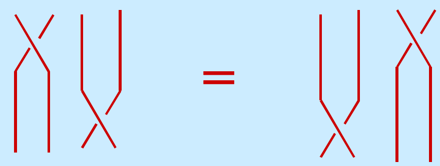
$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \;, 1 \le i < n$$

Reidemeister 3 Move



$$b_i b_j = b_j b_i \;, \text{for } |i-j| \ge 2$$

Planar Isotopy Move



A Presentation of the Braid Group B_n

$$\left(b_1, b_2, \ldots, b_{n-1} : \begin{array}{l} b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \;, 1 \le i < n-1 \\ b_i b_j = b_j b_i \;, |i-j| > 1, 1 \le i, j < n-1 \end{array} \right)$$

A Braid Is "Almost" a Permutation

$$B_n = \left(b_1, b_2, \ldots, b_{n-1} : \begin{array}{l} b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \;, 1 \le i < n-1 \\ b_i b_j = b_j b_i \;, |i-j| > 1, 1 \le i, j < n-1 \end{array} \right)$$

$\downarrow$ Natural Epimorphism

$$S_n = \left(b_1, b_2, \ldots, b_{n-1} : \begin{array}{l} b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \;, \quad 1 \le i < n-1 \\ b_i^2 = 1, \quad 1 \le i < n-1 \\ b_i b_j = b_j b_i \;, |i-j| > 1, \quad 1 \le i, j < n-1 \end{array} \right)$$

Why is the Braid Group Important ???

Why is the braid group important for Q Comp ?

- The representations of the Symmetric S_n are the basic building blocks for the representations of the unitary group U used in quantum mechanics,
- The braid group B_n "sits above" the symmetric group S_n , i.e., there is a natural epimorphism $B_n \downarrow S_n$
- Thus, new representations of the braid group B_n will give us new representations of the unitary group U, i.e., quantum gates
- **Claim**: These quantum gates can be implemented in quantum systems that are resistant to decoherence because of topological obstructions, e.g., in terms of the fractional quantum Hall effect, anyonic systems

Anyons: A Very Brief Overview

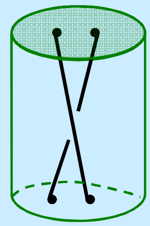
Anyons are quantum systems that are confined to two dimensions. They were first proposed by Nobel Laureate F. Wilczek. See for example,

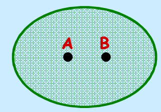
Wilczek, F., Fractional statistics and anyon superconductivity, World Scientific Press, (1990).

Anyons can used to explain the fractional quantum Hall effect

A Braid Represents the Movement of n Holes in a Disc

This braiding can be used to represent Anyon exchanges

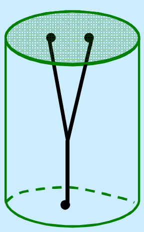


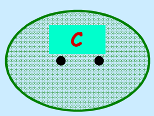


Anyonic braiding corresponds to a Unitary transformation

Recall: Q.M.= Qroup Rep. Theory

Anyons Can Also Fuse or Split





Anyons: A Very Brief Overview (Cont.)

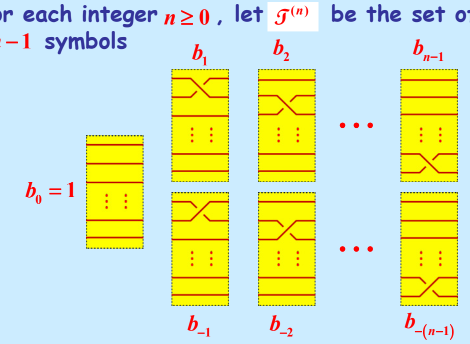
Quantum Topology gives us the tools needed to find new unitary representations based on fusing and braiding

These new unitary transformations are created with an object called a unitary topological modular functor which we call simply an anyon model.

Recall: Q.M.= Qroup Rep. Theory

Braid Mosaics

For each integer $n \geq 0$, let $\mathcal{T}^{(n)}$ be the set of $2n-1$ symbols



called braid n-stranded tiles, or simply tiles, and also respectively denoted by $b_0, b_{\pm 1}, b_{\pm 2}, \ldots, b_{\pm(n-1)}$

The Set $\mathcal{B}^{(n,\ell)}$ of Braid (n,ℓ)-Mosaics

Def. A braid (n,ℓ)- mosaic is a sequence of length ℓ

$$b_{j(1)}, b_{j(2)}, \ldots, b_{j(\ell)}$$

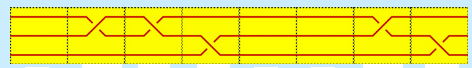
of braid n-tiles. Let $\mathcal{B}^{(n,\ell)}$ be the set of all braid (n,ℓ)-mosaics.

The Set $\mathcal{B}^{(n,\ell)}$ of Braid (n,ℓ)-Mosaics

Example: The braid (3,8)-mosaic

$$\beta = 1 b_{-1} b_1 b_2 1 1 b_{-1} b_2$$

is an element of $\mathcal{B}^{(3,8)}$.

$\beta =$ 

$1 \quad b_{-1} \quad b_1 \quad b_2 \quad 1 \quad 1 \quad b_{-1} \quad b_2$

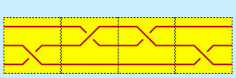
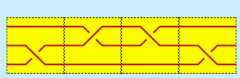
Observation: The cardinality of the set $\mathcal{B}^{(n,\ell)}$ of braid (n,ℓ)-mosaics is $(2n-1)^{\ell}$

Braid Mosaic Moves

Braid Moves for the set $\mathcal{B}^{(n,\ell)}$ of Braid (n,ℓ)-Mosaics

Def. A braid move on a braid mosaic β is a (cut & paste) operation that transforms β into another braid β' by replacing a submosaic of β by another. The location λ of the braid move is the location of the leftmost symbol in β effected by the move.

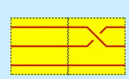
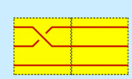
Example:

 λ=2 ⟷ 

The Planar Isotopy Moves

Move P_1 $\quad 1b_i \overset{\lambda}{\longleftrightarrow} b_i 1$ for $0 < |i| < n$

Observation: The number of P_1 moves is $2(n-1)(\ell-1)$

Example:  $\overset{\lambda}{\longleftrightarrow}$ 

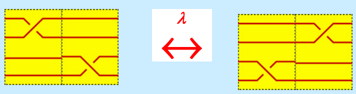
The Planar Isotopy Moves

Move P_2 $b_i b_j \overset{\lambda}{\longleftrightarrow} b_j b_i$ for $0<|i|,|j|<n$ & $\left||i|-|j|\right|>1$

Observation: The number of P_2 moves is

$$(n-1)(2n-6)(\ell-1)$$

Example:



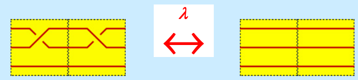
The Reidemeister Moves

Move R_2 $b_i b_{-i} \overset{\lambda}{\longleftrightarrow} 1^2$ for $0<|i|<n$

Observation: The number of R_2 moves is

$$2(n-1)(\ell-1)$$

Example:



The Reidemeister Moves

Move R_3 for $0<i<n$ or $-n<i<-1$

$$b_i b_{i+1} b_i b_{-(i+1)} b_{-i} b_{-(i+1)} \overset{\lambda}{\longleftrightarrow} 1^6$$

$$b_i b_{i+1} b_i b_{-(i+1)} b_{-i} \overset{\lambda}{\longleftrightarrow} b_{i+1} 1^4$$

$$b_i b_{i+1} b_i b_{-(i+1)} \overset{\lambda}{\longleftrightarrow} b_{i+1} b_i 1^2$$

$$b_i b_{i+1} b_i \overset{\lambda}{\longleftrightarrow} b_{i+1} b_i b_{i+1}$$

$$b_i b_{i+1} 1^2 \overset{\lambda}{\longleftrightarrow} b_{i+1} b_i b_{i+1} b_{-i}$$

$$b_i 1^4 \overset{\lambda}{\longleftrightarrow} b_{i+1} b_i b_{i+1} b_{-i} b_{-(i+1)}$$

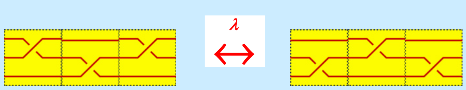
The Reidemeister Moves

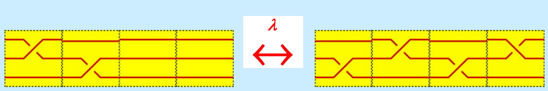
Observation: The number of R_3 moves is

$$\#R_3 Moves = \begin{cases} n(n-2)(6\ell-21) & \text{if } \ell \geq 6 \\ n(n-2)(5\ell-16) & \text{if } \ell = 5 \\ n(n-2)(3\ell-8) & \text{if } \ell = 4 \\ n(n-2)(\ell-2) & \text{if } \ell = 3 \\ 0 & \text{if } \ell < 3 \end{cases}$$

The Reidemeister R_3 Moves

Examples:





The Ambient Group

Braid Mosaic Moves Are Permutations

Each braid mosaic move acts as a local transf on an braid (n, ℓ)-mosaic whenever its conditions are met. If its conditions are not met, it acts as the identity transformation.

Ergo, each braid mosaic move is a permutation on the set $\mathcal{B}^{(n,\ell)}$ of all braid (n, ℓ)-mosaics

In fact, each braid mosaic move, as a permutation, is a product of disjoint transpositions.

The Ambient Group $A(n,\ell)$

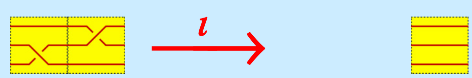
We define the ambient group $A(n,\ell)$ as the subgroup of the group of all permutations of the set $\mathcal{B}^{(n,\ell)}$ generated by the all braid (n,ℓ)-moves.

Braid Type

The Braid Mosaic Injection $\iota : \mathcal{B}^{(n,\ell)} \to \mathcal{B}^{(n,\ell+1)}$

We define the braid mosaic injection $\iota : \mathcal{B}^{(n,\ell)} \to \mathcal{B}^{(n,\ell+1)}$ as

$$\begin{array}{ccc} \mathcal{B}^{(n,\ell)} & \xrightarrow{\iota} & \mathcal{B}^{(n,\ell+1)} \\ \beta = b_{j(1)} b_{j(2)} \cdots b_{j(\ell)} & \mapsto & \beta' = b_{j(1)} b_{j(2)} \cdots b_{j(\ell)} 1 \end{array}$$



Mosaic Braid Type

Def. Two braid (n,ℓ)-mosaics β and β' are of the same braid (n,ℓ)-mosaic type, written

$$\beta \underset{n}{\overset{\ell}{\sim}} \beta'$$

provided there exists an element of the ambient group $A(n,\ell)$ that transforms β into β'.

Two (n,ℓ)-mosaics β and β' are of the same braid type if there exists a non-negative integer k such that

$$\iota^k \beta \underset{n}{\overset{\ell+k}{\sim}} \iota^k \beta'$$

Part 2

Quantum Braids & Quantum Braid Systems

The Hilbert Space $\widehat{\mathcal{B}}^{(n,\ell)}$ of (n,ℓ)-mosaics

Let $\mathcal{H}^{(n)}$ be the **2n-1** dimensional Hilbert space with orthonormal basis labeled by the tiles

$$b_0, b_{\pm 1}, b_{\pm 2}, \ldots, b_{\pm(n-1)}$$

We define the Hilbert space $\mathcal{H}^{(n,\ell)}$ of braid **(n,ℓ)**- mosaics as

$$\widehat{\mathcal{B}}^{(n,\ell)} = \bigotimes_{k=1}^{\ell} \mathcal{H}^{(n)}$$

This is the Hilbert space with induced orthonormal basis

$$\left\{ \bigotimes_{k=1}^{\ell} \left| b_{j(k)} \right\rangle \; : \; -n < j(k) < n \right\}$$

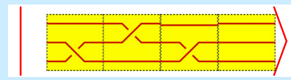
The Hilbert Space $\widehat{\mathcal{B}}^{(n,\ell)}$ of Braid (n,ℓ)-Mosaics

We identify each basis ket $\bigotimes_{k=1}^{\ell} \left| b_{j(k)} \right\rangle$ with a ket $\left| \beta \right\rangle$ labeled by a braid **(n,ℓ)**-mosaic β.

For example, in the braid **(3,4)**-mosaic Hilbert space $\widehat{\mathcal{B}}^{(3,4)}$, the basis ket

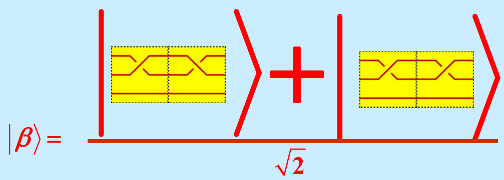
$$\left| b_2 \right\rangle \otimes \left| b_{-1} \right\rangle \otimes \left| b_{-2} \right\rangle \otimes \left| b_0 \right\rangle$$

is identified with braid **(3,4)**-mosaic labeled ket



A quantum braid is an element of $\widehat{\mathcal{B}}^{(3,4)}$

An Example of a Quantum Braid



A quantum braid (3,2)-mosaic

The Ambient Group A(n,ℓ) as a Unitary Group

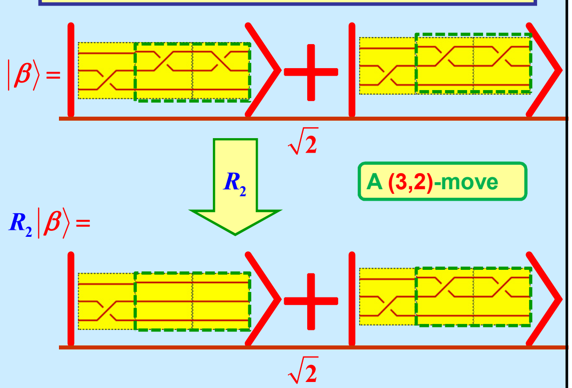
We identify each element $g \in A(n,\ell)$ with the linear transformation defined by

$$\begin{array}{ccc} \widehat{\mathcal{B}}^{(n,\ell)} & \to & \widehat{\mathcal{B}}^{(n,\ell)} \\ \left|\beta\right\rangle & \mapsto & \left|g\beta\right\rangle \end{array}$$

Since each element $g \in A(n,\ell)$ is a permutation, it is a linear transformation that simply permutes basis elements.

Hence, under this identification, the ambient group $A(n,\ell)$ becomes a discrete group of unitary transfs on the Hilbert space $\widehat{\mathcal{B}}^{(n,\ell)}$.

An Example of the $A(n,\ell)$ Group Action



The Quantum Braid System $\left(\widehat{\mathcal{B}}^{(n,\ell)}, A(n,\ell)\right)$

Def. A quantum braid system $\left(\widehat{\mathcal{B}}^{(n,\ell)}, A(n,\ell)\right)$ is a quantum system having $\widehat{\mathcal{B}}^{(n,\ell)}$ as its state space, and having the Ambient group $A(n,\ell)$ as its set of accessible unitary transformations.

The states of quantum system $\left(\widehat{\mathcal{B}}^{(n,\ell)}, A(n,\ell)\right)$ are quantum braids. The elements of the ambient group $A(n,\ell)$ are quantum moves.

Quantum Braid Type

Def. Two quantum braid (n,ℓ)-mosaics $|\beta_2\rangle$ and $|\beta_1\rangle$ are of the same braid (n,ℓ)-type, written

$$|\beta_1\rangle \underset{n}{\overset{\ell}{\sim}} |\beta_2\rangle,$$

provided there is an element $g \in A(n,\ell)$ s.t.

$$g|\beta_1\rangle = |\beta_2\rangle$$

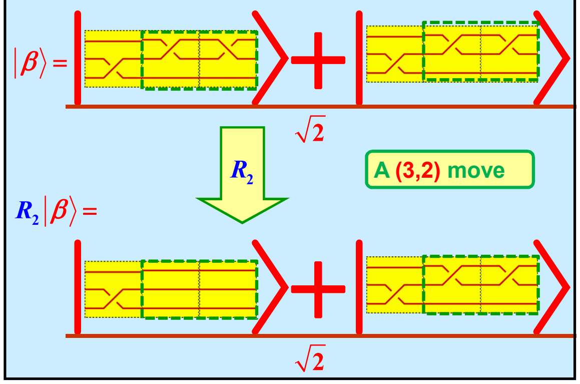
They are of the same braid type, written

$$|\beta_1\rangle \sim |\beta_2\rangle,$$

provided there is an integer $m \geq 0$ such that

$$\iota^m|\beta_1\rangle \underset{n}{\overset{\ell+m}{\sim}} \iota^m|\beta_2\rangle$$

Two Quantum Braids of the Same Braid Type



Hamiltonians of the Generators of the Ambient Group

Hamiltonians for $A(n)$

Each generator $g \in A(n,\ell)$ is the product of disjoint transpositions, i.e.,

$$g = (K_{\alpha_1}, K_{\beta_1})(K_{\alpha_2}, K_{\beta_2})\cdots(K_{\alpha_\ell}, K_{\beta_\ell})$$

Choose a permutation η so that

$$\eta^{-1}g\eta = (K_1, K_2)(K_3, K_3)\cdots(K_{\ell-1}, K_\ell)$$

Hence,

$$\eta^{-1}g\eta = \begin{pmatrix} \sigma_1 & & & & \mathbf{0} \\ & \sigma_1 & & & \\ & & \ddots & & \\ & & & \sigma_1 & \\ \mathbf{0} & & & & I_{n-2\ell} \end{pmatrix}$$

, where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Hamiltonians for $A(n,\ell)$

Also, let $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and note that

$$\ln(\sigma_1) = \frac{i\pi}{2}(2s+1)(\sigma_0 - \sigma_1), \quad s \in \mathbb{Z}$$

For simplicity, we always choose the branch $s = 0$

$$H_g = -i\eta \ln(\eta^{-1}g\eta)\eta^{-1}$$

$$= \frac{\pi}{2}\eta \begin{pmatrix} I_\ell \otimes (\sigma_0 - \sigma_1) & 0 \\ 0 & 0_{(n-2\ell)\times(n-2\ell)} \end{pmatrix} \eta^{-1}$$

Observables which are Quantum Braid Invariants

Observable Q. Braid Invariants

Question. What do we mean by a physically observable braid invariant ?

Let $\left(\widehat{\mathcal{B}}^{(n,\ell)}, A(n,\ell)\right)$ be a quantum braid system. Then a quantum observable Ω is a Hermitian operator on the Hilbert space $\widehat{\mathcal{B}}^{(n,\ell)}$.

Observable Q. Braid Invariants

Question. But which observables Ω are actually braid invariants ?

Def. An observable Ω is an invariant of quantum braids provided $U\Omega U^{-1} = \Omega$ for all $U \in A(n,\ell)$

Observable Q. Knot Invariants

Question. But how do we find quantum braid invariant observables ?

Theorem. Let $\left(\widehat{\mathcal{B}}^{(n,\ell)}, A(n,\ell)\right)$ be a quantum braid system, and let

$$\widehat{\mathcal{B}}^{(n,\ell)} = \oplus_j W_j$$

be a decomposition of the representation $A(n,\ell) \times \widehat{\mathcal{B}}^{(n,\ell)} \to \widehat{\mathcal{B}}^{(n,\ell)}$ into irreducible representations .

Then, for each j, the projection operator P_j for the subspace W_j is a quantum braid observable.

Observable Q. Braid Invariants

Theorem. Let $\left(\widehat{\mathcal{B}}^{(n,\ell)}, A(n,\ell)\right)$ be a quantum braid system, and let Ω be an observable on $\widehat{\mathcal{B}}^{(n,\ell)}$. Let $St(\Omega)$ be the stabilizer subgroup for Ω , i.e.,

$$St(\Omega) = \left\{ U \in A(n,\ell) : U\Omega U^{-1} = \Omega \right\}$$

Then the observable

$$\sum_{U \in A(n)/St(\Omega)} U\Omega U^{-1}$$

is a quantum braid invariant, where the above sum is over a complete set of coset representatives of $St(\Omega)$ in $A(n,\ell)$.

Future Directions & Open Questions

Future Directions & Open Questions

- Presentation of the ambient group A(n,ℓ)
- How is the ambient group A(n,ℓ) related to the homology group of the braid group?
- Can quantum braids be used to simplify the search for unitary representations of the braid group?

Future Directions & Open Questions

The Yang-Baxter relation "lives" in the ambient group A(n,ℓ) . Can it be lifted to the Lie algebra of the unitary group $\mathcal{U}\left(\widehat{\mathcal{B}}^{(n,\ell)}\right)$?

If so, the search for unitary reps of the braid group reduces to the task of associating Hamiltonians with the generators of the braid group.

Future Directions & Open Questions

If so, we could choose an assignment of Hamiltonians

$$H_j \leftrightarrow g_j$$

which is consistent with the Yang-Baxter relation.

These Hamiltonians determine a unitary evolution of Schroedinger's equation, which is a unitary representation of the braid group.

Future Directions & Open Questions

As an example, we have found Hamiltonians that produce the Fibonnacci representation.

Question: Can we find a general way to lift the Yang-Baxter relation?