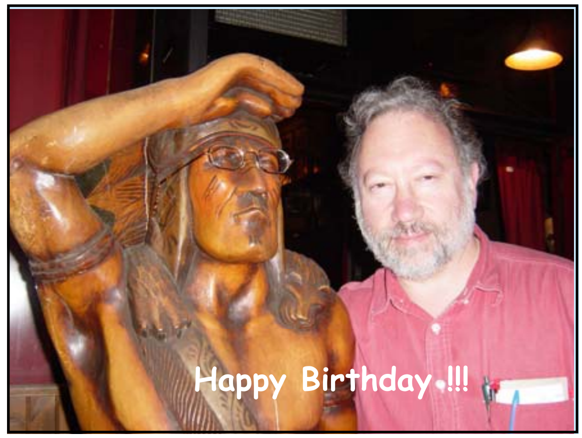

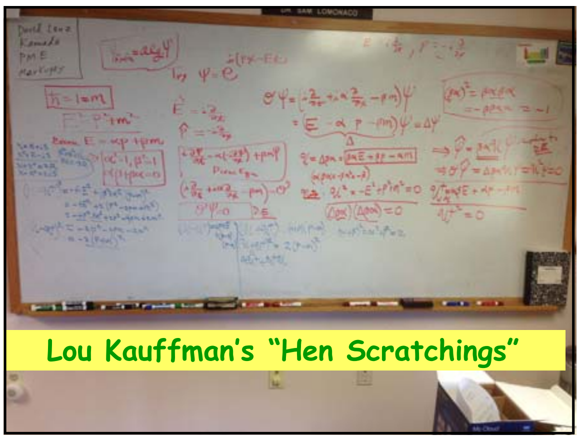

The Geometry of Fox's Free Calculus with Applications to Higher Dimensional Knots

Work in Progress

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Talk given at Knots in Hellas, July 19, 2016

PowerPoint slides can be found at:
www.csee.umbc.edu/~lomonaco/Lectures.html

The Fox Free Calculus

Algebraic Def. of Fox Free Derivative $\partial / \partial x_j$

Let G be a group, and let $\mathbb{Z}G$ denote the corresponding group ring over the ring of integers $\mathbb{Z}$.

Def. A derivative D on a group ring $\mathbb{Z}G$ is defined as a map

$$D : \mathbb{Z}G \to \mathbb{Z}G$$

satisfying the following condition:

1) $D(\omega_1 + \omega_2) = D\omega_1 + D\omega_2$
2) $D(\omega_1\omega_2) = (D\omega_1)\omega_2^o + \omega_1 D\omega_2$

where $^o : \mathbb{Z}G \to \mathbb{Z}$ is the trivializer morphism which maps each element of G to 1 of $\mathbb{Z}$.

Algebraic Def. of Fox Free Derivative $\partial / \partial x_j$

Def.(Cont.) Let G be the free group $F(\underline{\underline{x}})$. Then to each free generator $x_j \in \underline{\underline{x}}$, there corresponds a unique derivative

$$D_j = \partial / \partial x_j$$

in $\mathbb{Z}F(\underline{\underline{x}})$, called the derivative with respect to x_j , which has the property

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \quad \text{(Kronecker delta)}$$

Immediate consequence of the def of $\partial / \partial x_j$

Let

$$w_1, w_2 \in F(\underline{\underline{x}})$$

then

$$\frac{\partial}{\partial x_j}(w_1 w_2) = \frac{\partial(w_1)}{\partial x_j} + w_1 \frac{\partial(w_2)}{\partial x_j}$$

But Where is the Geometry ???

The Search for Geometry

Knot Theory

Various Placement Problems

- 3-D Knot Theory
  $k : S^1 \to S^3$ — 1-Knot $\left(S^3, kS^1\right)$
- 4-D Knot Theory
  $k : S^2 \to S^4$ — 2-Knot $\left(S^4, kS^2\right)$
- 5-D Knot Theory
  $k : S^3 \to S^5$ — 3-Knot $\left(S^5, kS^3\right)$

3-D Knot Theory

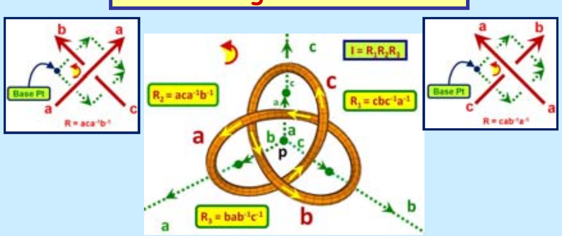
1-Knots

The Fundamental Group $\pi_1(X)$

Knot Exterior $X = S^3 - SmallOpenTubularNbd\left(kS^1\right)$

- Fundamental Group $\pi_1(X) =$ Knot Invariant
- Asphericity of Knots (Papakyriakopoulos)
  $\Rightarrow X = K(\pi_1 X, 1). \quad \therefore \pi_n X = 0 \ for \ n > 1$

The Wirtinger Presentation



$$\left(\underbrace{a, b, c}_{\text{Generators}} : \underbrace{r_1, r_2, r_3}_{\text{Relators}}\right)$$

Generators | Relators

Group Presentations

__Question__: When do two presentations represent the same group ?

Tietze Transformations: $T_1^{\pm 1}, T_2^{\pm 1}$

Tietze 1: $\left(\underline{\underline{x}} : \underline{\underline{r}}\right) \xrightarrow{T_1} \left(\underline{\underline{x}} \cup y : \underline{\underline{r}} \cup s\right)$, where

- y is a new symbol, and
- $s = y\xi^{-1}$, with $\xi \in F\left(\underline{\underline{x}}\right)$

Tietze 2: $\left(\underline{\underline{x}} : \underline{\underline{r}}\right) \xrightarrow{T_2} \left(\underline{\underline{x}} : \underline{\underline{r}} \cup s\right)$, where

- $s \in Cons\left(\underline{\underline{r}}\right)$, i.e., $s = \prod_{\alpha=1}^{m} r_{j(\alpha)}^{w_\alpha}$, with
- $w_\alpha \in F\left(\underline{\underline{x}}\right)$, $0 \le \alpha \le m$, and
- $r_{j(\alpha)}^{w_\alpha} = w_\alpha \cdot r_{j(\alpha)} \cdot w_\alpha^{-1}$

Group Presentations

Question: When do two presentations represent the same group ?

Theorem (Tietze): Two group presentations represent the same group iff there exists a finite sequence of Tietze transformations which transforms one into the other.

The Geometry of Group Presentations

Def. An abstract Group presentation consists of two sets

- $\underline{\underline{x}}$, the set of generators,
- $\underline{\underline{r}}$, the set of relators,

together with an evaluation map

$$\wedge : \underline{\underline{r}} \to F\left(\underline{\underline{x}}\right)$$

from the set of relators $\underline{\underline{r}}$ into the free group $F\left(\underline{\underline{x}}\right)$ on the set of symbols $\underline{\underline{x}}$.

The Geometry of Group Presentations

Example: $\left(a,b,c : r_1, r_2, r_3\right)$, where

$$\begin{cases} \hat{r}_1 = cbc^{-1}a^{-1} \\ \hat{r}_2 = aca^{-1}b^{-1} \\ \hat{r}_3 = bab^{-1}c^{-1} \end{cases}$$

The Geometry of Group Presentations

Def. A CW-complex is said to be monopointed it it has only one 0-cell.

Proposition. Up to renaming and reordering, there exists a one-to-one correspondence between the set of abstract group presentations and a set of monopointed 2-D CW-complexes.

$$\left(\underline{\underline{x}} : \underline{\underline{r}}\right) \leftrightarrow K\left(\underline{\underline{x}} : \underline{\underline{r}}\right)$$

The Geometry of Group Presentations

The CW-complex $K\left(\underline{\underline{x}} : \underline{\underline{y}}\right)$ is constructed with an initial 0-cell, denoted by ∞ , and then iteratively attaching cells as follows:

1-cells: For each generator $x_j \in \underline{\underline{x}}$, adjoin an oriented 1-cell X_j by attaching both endpoints to the sole 0-cell ∞ .

2-cells: For each relator $r_k \in \underline{\underline{r}}$, adjoin an oriented 2-cell r_k with attaching map $\hat{r}_k$.

Examples

Example: $Torus = (a,b:r), \hat{r} = ab\bar{a}\bar{b}$

Example: $\mathbb{R}P^2 = (a,b:r), \hat{r} = aba\bar{b}$

Example: $S^1 \vee D^2 = (a,b:r), \hat{r} = b$

Example: $S^1 \vee S^1 \vee S^2 = (a,b:r), \hat{r} = 1$

Example: $S^2 \vee S^2 = (\varnothing : r_1, r_2), \hat{r_1} = 1, \hat{r_2} = 1$

The Geometry of the Tietze Moves

- Tietze 1 attaches a 2-cell S and a free edge Y .

$$T_1 : (\underline{\underline{x}} : \underline{\underline{r}}) \mapsto (\underline{\underline{x}} \cup y : \underline{\underline{r}} \cup s), \hat{s} = y\xi^{-1}, \xi \in F(\underline{\underline{x}})$$

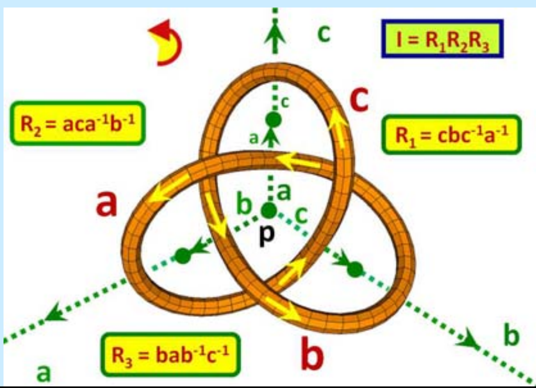
Thus, T_1 is a simple homotopy operation; and therefore preserves homotopy type.

- Tietze 2 attaches a 2-cell S.

$$T_2 : (\underline{\underline{x}} : \underline{\underline{r}}) \mapsto (\underline{\underline{x}} : \underline{\underline{r}} \cup s), \hat{s} \in Cons(\underline{\underline{r}})$$

Thus, T_2 does NOT necessarily preserve homotopy type !

The Geometry of the Wirtinger Presentation



A Paradox ?

Paradox: The Wirtinger presentation actually represents the 3-D knot exterior X as a 2-D CW complex. The sole 3-cell in the exterior X has been omitted !!!

The usual "fix": The Wirtinger presentation has one too many generators. So an unnecessary relator is tossed out by applying a Tietze 2 move.

Fortunately, in this particular case, the Tietze 2 move preserves the homotopy type because there is a simpe homotopy type move on the 3-D complex collapsing the 3-D complex to the same resulting 2-D complex

Observation

The 3-cell in X corresponds to an identity I among the relators, namely:

$$I \xrightarrow{\wedge} \hat{I} = r_1 r_2 r_3 \xrightarrow{\wedge} cb\bar{c}\bar{a} \cdot ac\bar{a}\bar{b} \cdot ba\bar{b}\bar{c} = 1$$

Where does this identity "live" ?

Groups with Operators

Def. Let H and G be groups. The group G is said to be an H-group provided there exists a morphism $H \to Aut(G)$ of H into the group Aut(G) of automorpisms of G .

Free $F(\underline{\underline{x}})$-groups

Def. Let $\underline{\underline{x}}$ and $\underline{\underline{t}}$ be two disjoint sets of symbols, and let $F(\underline{\underline{x}})$ and $F(\underline{\underline{x}} \cup \underline{\underline{t}})$ denote the corresponding free groups, respectively. The free $F(\underline{\underline{x}})$-group on the symbols $\underline{\underline{t}}$, written

$$\mathfrak{F}_{F(\underline{\underline{x}})}(\underline{\underline{t}}),$$

is the smallest normal subgroup of $F(\underline{\underline{x}} \cup \underline{\underline{t}})$ containing $\underline{\underline{t}}$.

It immediately follows that $\mathfrak{F}_{F(\underline{\underline{x}})}(\underline{\underline{t}})$ is invariant under the conjugation action of $F(\underline{\underline{x}})$.

Free $F(\underline{\underline{x}})$-groups

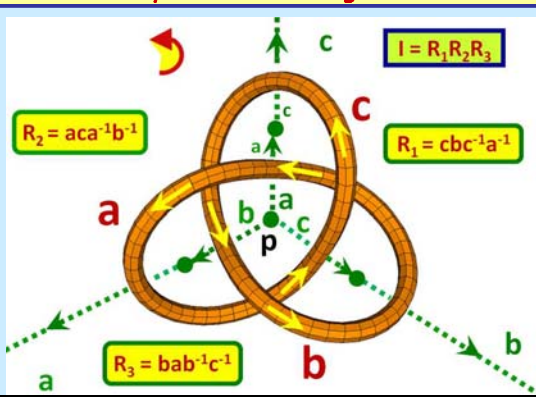
Thus, the elements of $\mathfrak{F}_{F(\underline{\underline{x}})}(\underline{\underline{t}})$ are of the form:

$$\prod_{\alpha} t_{j(\alpha)}^{w_\alpha}$$

where $t_{j(\alpha)}^{w_\alpha} = w_\alpha \cdot t_{j(\alpha)} \cdot w_{j(\alpha)}^{-1}$

Note. This conjugation action is a left action.

The Geometry of the Wirtinger Presentation



Wirtinger Hyper Presentation

The Wirtinger 3-D CW decomposition of the exterior X is nothing more than the non-abelian free resolution:

$$G \xleftarrow{\nu} F(a,b,c) \xleftarrow{\wedge} \mathfrak{F}_F(r_1,r_2,r_3) \xleftarrow{\wedge} \mathfrak{F}_F(I)$$

$$cb\bar{c}\bar{a} \xleftarrow{\wedge} r_1 \qquad r_1 r_2 r_3 \xleftarrow{\wedge} I$$

$$ac\bar{a}\bar{b} \xleftarrow{\wedge} r_2$$

$$ba\bar{b}\bar{c} \xleftarrow{\wedge} r_3$$

Please note that $\wedge^2 = 1$ and $\nu \wedge = 1$.

Wirtinger Hyper Presentation

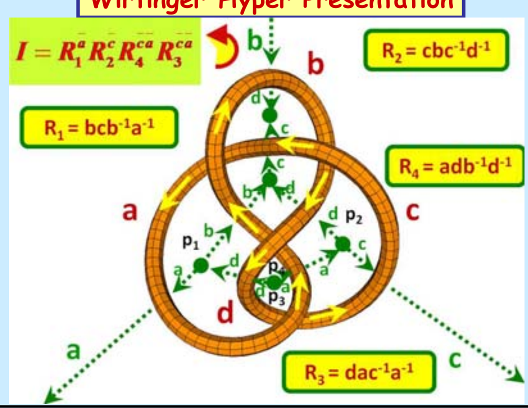
$$G \xleftarrow{\nu} F(a,b,c) \xleftarrow{\wedge} \mathfrak{F}_F(r_1,r_2,r_3) \xleftarrow{\wedge} \mathfrak{F}_F(I)$$

We denote the above non-abelian free resolution more cryptically as

$$(a,b,c : r_1,r_2,r_3 : I)$$

and call it a hyper presentation.

Wirtinger Hyper Presentation



Wirtinger Hyper Presentation

$$\left(a,b,c,d : r_1,r_2,r_3,r_3 : I\right)$$

$bc\bar{b}\bar{a} \overset{\wedge}{\longleftarrow} r_1$

$cb\bar{c}\bar{d} \overset{\wedge}{\longleftarrow} r_2$

$da\bar{c}\bar{a} \overset{\wedge}{\longleftarrow} r_3$

$ad\bar{b}\bar{d} \overset{\wedge}{\longleftarrow} r_4$

$r_1^a r_2^c r_4^{ca} r_3^{ca} \overset{\wedge}{\longleftarrow} I$

Terminology for Hyper Presentations

The current names identities, identities of identities, identities of identities of identities, etc. are too cumbersome. So we are forced to adopt the following terminology:

$$\left(\underline{\underline{x}} : \underline{\underline{r}}^{(2)} : \underline{\underline{r}}^{(3)} : \underline{\underline{r}}^{(4)} : \cdots\right)$$

- $\underline{\underline{x}}$: generators or 1-st order relators
- $\underline{\underline{r}}^{(2)}$: 2-nd order relators
- $\underline{\underline{r}}^{(3)}$: 3-rd order relators
- $\underline{\underline{r}}^{(4)}$: 4-th order relators

Hyper Tietze Moves

There is only one hyper Tietze move for each order, namely:

n-th order hyper Tietze move : T_n

$$T_n : \left(\cdots : \underline{\underline{r}}^{(n)} : \underline{\underline{r}}^{(n+1)} : \cdots\right) \mapsto \left(\cdots : \underline{\underline{r}}^{(n)} \cup \sigma : \underline{\underline{r}}^{(n+1)} \cup \tau : \cdots\right)$$

where $\hat{\tau} = \sigma\xi^{-1}$ and $\hat{\sigma} = \hat{\xi}$

and where $\xi \in \mathfrak{F}_F\left(\underline{\underline{r}}^{(n)}\right)$

Hyper Presentation Equivalence

The definition of hyper presentation equivalence is a straight forward exercise for the audience. So is the proof of the following theorem:

Theorem: Two hyper presentations are equivalent iff there is a finite sequence of hyper Tietze moves that transform one into the other.

The Geometry of Hyper Presentations

Proposition. Up to renaming and reordering, there exists a one-to-one correspondence between the set of finite hyper presentations and a set of monopointed CW-complexes.

$$\left(\underline{\underline{x}} : \underline{\underline{r}} : \underline{\underline{r}}^{(3)} : \underline{\underline{r}}^{(4)} : \cdots\right) \leftrightarrow K\left(\underline{\underline{x}} : \underline{\underline{r}} : \underline{\underline{r}}^{(3)} : \underline{\underline{r}}^{(4)} : \cdots\right)$$

The hyper Tietze moves correspond to **simple homotopy** moves on the associated CW-complexes.

Moreover, two hyper presentations define CW complexes of the same **simple homotopy type** iff there is a finite sequence of Tieze moves which transforms one into the other.

The Geometry of the Fox Free Calculus

Let $\mathfrak{P} = \left(\underline{\underline{x}} : \underline{\underline{r}}^{(2)} : \underline{\underline{r}}^{(3)} : \cdots : \underline{\underline{r}}^{(n)}\right)$ be a hyper presentation, and let

$$K = K\left(\mathfrak{P}\right)$$

be the corresponding CW-complex.

Let $G = \pi_1\left(K\right)$ the fundamental group of K.

Let $\nu : F\left(\underline{\underline{x}}\right) \to G$ be the epimorphism associated with $\left(\underline{\underline{x}} : \underline{\underline{r}}^{(2)}\right)$

Finally, let $\mathbb{Z}G$ be the group ring of G over the integers $\mathbb{Z}$.

The Geometry of the Fox Free Calculus

Let $\widetilde{K}$ be the universal cover of K, and let $\widetilde{\widetilde{K}} = \widetilde{K} \times Ker\,\nu$ be the non pathwise connected space above $\widetilde{K}$.

We now use the Fox free derivatives to construct a chain complex $C_* = C_*\left(\widetilde{\widetilde{K}}\right)$

Hence, $H_*\left(\widetilde{\widetilde{K}}\right) = H_* C_*$

Moreover, $H_*\left(\widetilde{K}\right) = H_*\left(\mathbb{Z}G \otimes_{\mathbb{Z}F} C_*\right)$

The Geometry of the Fox Free Calculus

The chain groups are defined as follows:

- The 0-th chain group $C_0 = C_0(\infty)$ is defined as the free $\mathbb{Z}F$ -module generated by the 0-cell ∞ .
- For n > 0 , the n-th chain group $C_n = C_n\left(\underline{\underline{R}}^{(n)}\right)$ is defined as the free $\mathbb{Z}F$ -module generated by the set of n-cells $\underline{\underline{R}}^{(n)}$.

The Geometry of the Fox Free Calculus

The boundary morphisms are defined as follows:

- For n = 0 ,

$$C_0(\infty) \xleftarrow{\partial} C_1\left(\underline{\underline{X}}\right)$$

$$\left(x_j - 1\right)\infty \leftarrow X_j$$

- For n > 0 ,

$$C_{n-1}\left(\underline{\underline{R}}^{(n-1)}\right) \xleftarrow{\partial} C_n\left(\underline{\underline{R}}^{(n)}\right)$$

$$\sum_k \left(\frac{\partial \hat{r}_j^{(n)}}{\partial r_k^{(n-1)}}\right) R_k^{(n-1)} \leftarrow R_k^{(n)}$$

Where the Fox Free derivatives $\partial / \partial x_j$ are geometrically defined as follows:

Cell Decompositions of $K, \widetilde{K}, \widetilde{\widetilde{K}}$

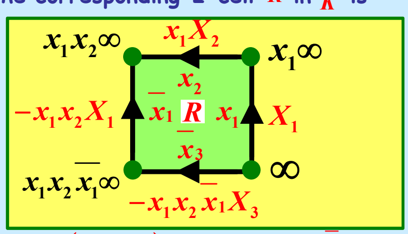
Epimorphism: $F\left(\underline{\underline{x}}\right) \xrightarrow{\nu} G = \pi_1 K$

$\widetilde{K} \times Ker\,\nu =$	$\widetilde{\widetilde{K}}$	$\{w\infty\}$	$\{wX_i\}$	$\{wR_j\}$	$\{wR_k^{(3)}\}$	$\{wR_4^{(4)}\}$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
Univ. Cover	$\widetilde{K}$	$\{g\infty\}$	$\{gX_i\}$	$\{gR_j\}$	$\{gR_k^{(3)}\}$	$\{gR_4^{(4)}\}$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
CW-Complex	K	$\{\infty\}$	$\{X_i\}$	$\{R_j\}$	$\{R_k^{(3)}\}$	$\{R_\ell^{(4)}\}$
		0-cells	1-cells	2-cells	3-cells	4-cells

$g \in G$ $\quad w \in F\left(\underline{\underline{x}}\right)$

Recall $\mathfrak{P} = \left(\underline{\underline{x}} : \underline{\underline{r}}^{(2)} : \underline{\underline{r}}^{(3)} : \cdots : \underline{\underline{r}}^{(n)}\right)$ and $K = K(\mathfrak{P})$.

If, for example, $r \in \underline{\underline{r}}^{(2)}$ with $\hat{r} = x_1 x_2 \bar{x}_1 \bar{x}_3$, then the corresponding 2-cell R in $\widetilde{\widetilde{K}}$ is



$$\partial R = \left(1 - x_1 x_2\right) X_1 + x_1 X_2 - x_1 x_2 \bar{x}_1 X_3$$

$$= \frac{\partial \hat{r}}{\partial x_1} X_1 + \frac{\partial \hat{r}}{\partial x_2} X_2 + \frac{\partial \hat{r}}{\partial x_3} X_3$$

If, for example, $u \in \underline{\underline{r}}^{(3)}$ with $\hat{u} = r_1^{x_1} r_2^{x_3} \bar{r}_1^{x_1 x_3}$, then the boundary chain map of the corresponding 3-cell U in $\tilde{\tilde{\tilde{K}}}$ is:

$$\partial U = \left(x_1 - \hat{r}_1^{\,x_1} \hat{r}_2^{\,x_3} x_1 x_3 \hat{\bar{r}} \right) R_1 + \hat{r}_1^{\,x_1} x_3 R_2$$

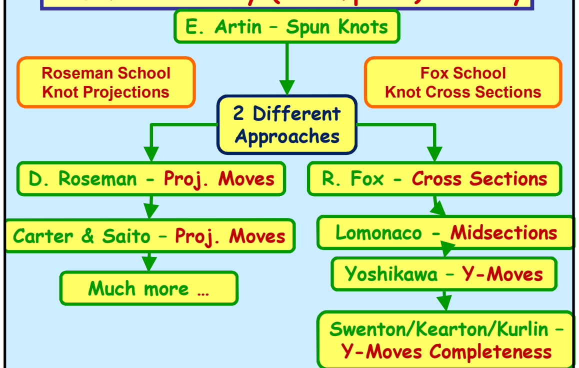
$$= \left(\frac{\partial \hat{u}}{\partial r_1} \right)^{\wedge} R_1 + \left(\frac{\partial \hat{u}}{\partial r_2} \right)^{\wedge} R_2$$

2-Knots

Various Placement Problems

- 3-D Knot Theory
  $k : S^1 \to S^3$ — 1-Knot (S^3, kS^1)
- 4-D Knot Theory
  $k : S^2 \to S^4$ — 2-Knot (S^4, kS^2)
- 5-D Knot Theory
  $k : S^3 \to S^5$ — 3-Knot (S^5, kS^3)

4-D Knot Theory (Incomplete) History



4-D Analog of the Apspericity of Knots ?

For 1-knots, asphericity implies the exterior X is an Eilenberg-MacLane space, i.e.,

$$X = K(\pi_1 X, 1)$$

Question. Want can be said about analog of Papa's asphericity theorem for 2-knots?

4-D Analog of the Apspericity of Knots ?

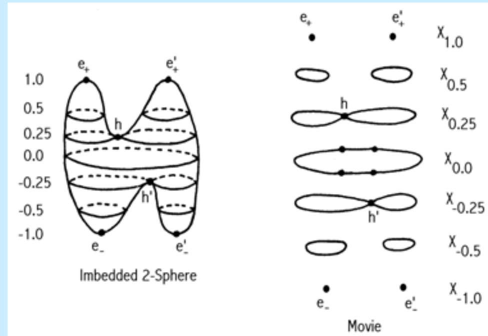
Def. A 2-knot (S^4, kS^2) is said to be quasi-aspherical (QA) if the third homology group of the universal cover of its exterior vanishes.

Theorem. (Lomonaco) If (S^4, kS^2) is QA, then the homotopy type of its exterior X is determined by its algebraic 3-type, i.e., by the triple consisting of:

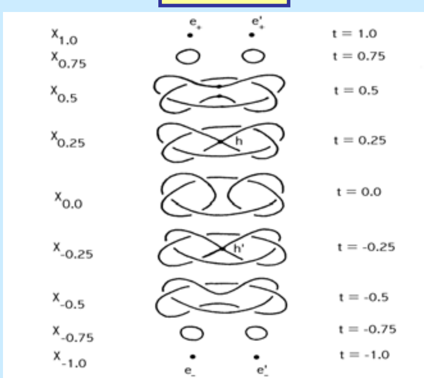
- $\pi_1 X$
- $\pi_2 X$ as a $\mathbb{Z}\pi_1 X$ -module
- The first k-invariant kX lying in $H^3(\pi_1 X; \pi_2 X)$

The Cross sectional Approach to 2-Knots

Edwin Abbott's Flatland



3-D Land

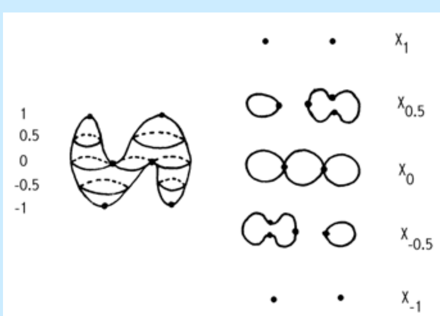


The Midsection Representation 2-Knots

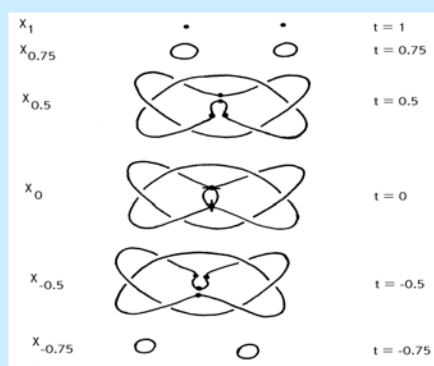
Reducing

4-D Knot Theory
to
3-D Knot Theory

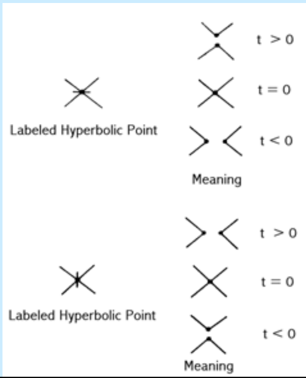
Edwin Abbott's Flatland



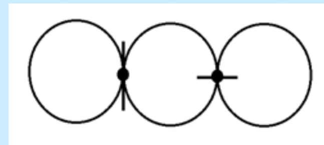
3-D Land



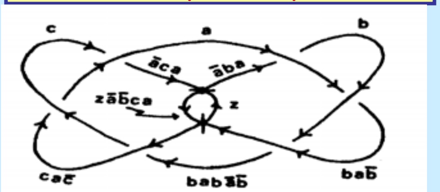
Midsection Labeling Scheme



Midsection Representation of a 2-sphere

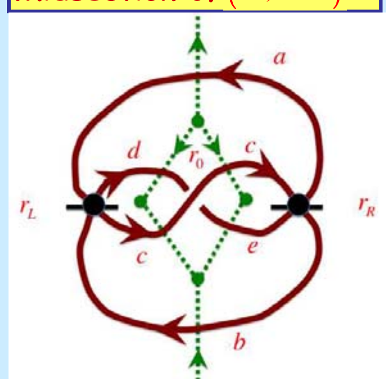


Midsection Rep of a Spun Trefoil



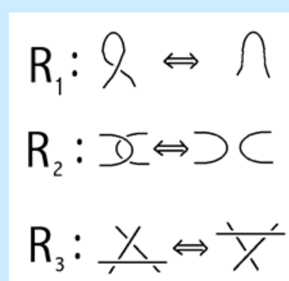
$$\begin{cases} \pi_1 K = (\ a, b : abab^{-1}a^{-1}b^{-1} = 1\ ) \\ \pi_2 K = (\ \partial U_{\square} : (1-a+ba)\,\partial U_{\square} = 0\ ) \\ H^3(\pi_1 K; \pi_2 K) = 0 \quad \text{and} \quad \underline{k}K = 0 \end{cases}$$

Midsection of $(S^4, k\mathbb{R}P^2)$

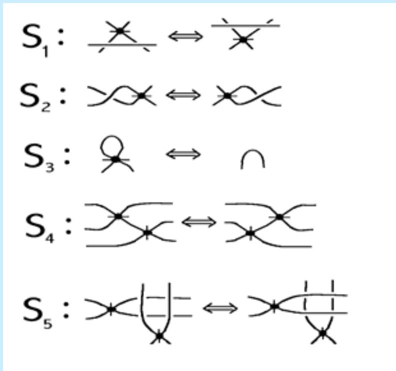


Midsection Moves on 2-Knots

Reidemeister Moves



Yoshikawa Moves



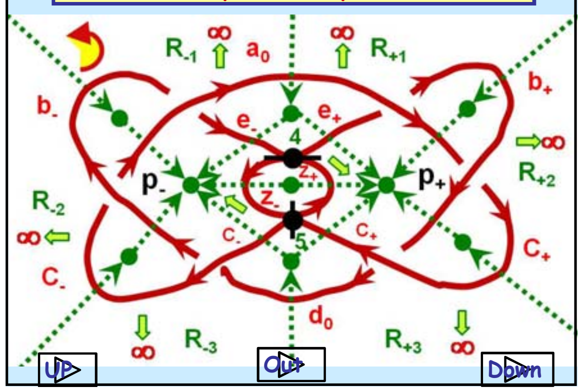
The Geometry of Group Presentations

Theorem(Yoshikawa). The Reidemeister and Yoshikawa moves on the midsection representation of a 2-knot preserve knot type.

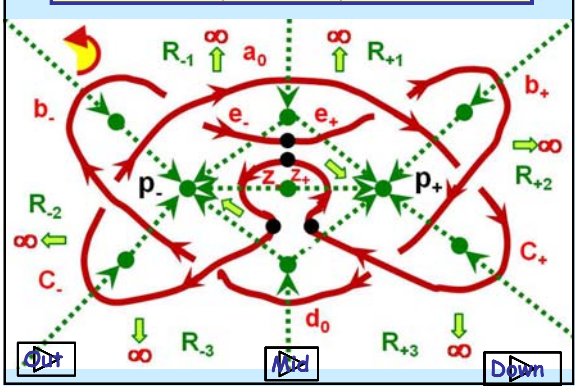
Theorem(Swenton/Keartin/Kurlin). Two 2-knot midsections represent the same 2-knot iff there is a finite sequence of Reidemeister and Yoshikawa moves which transforms one into the other.

The Wirtinger Hyper Presentation of 2-Knots ???

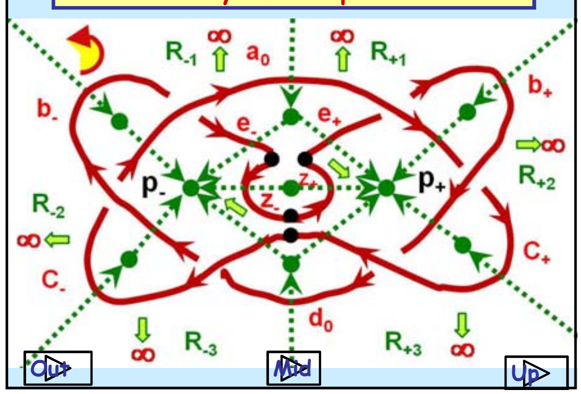
The Geometry of Group Presentations



The Geometry of Group Presentations

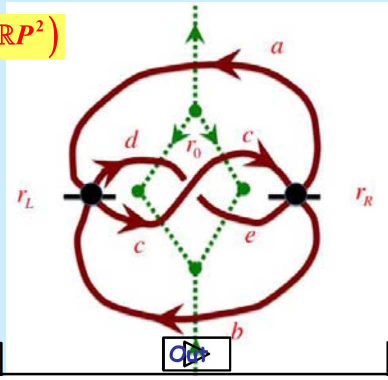


The Geometry of Group Presentations



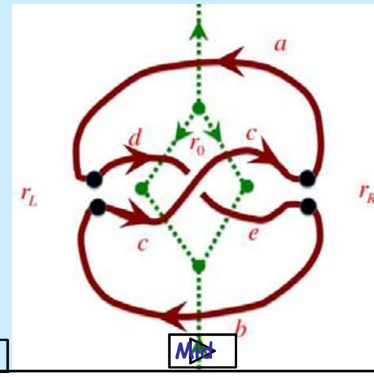
The Geometry of Group Presentations

$\left(S^4, \mathbb{R}P^2\right)$



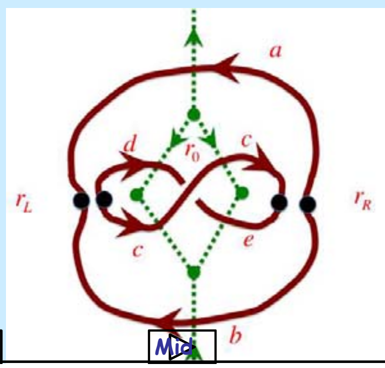
UP Out Down

The Geometry of Group Presentations



Out Mid Down

The Geometry of Group Presentations



UP Mid Out

Ongoing Research Program

This talk is a description of an ongoing research program which is best summarized by the following two questions:

Objective 1

Let X be the exterior of a 2-knot $\left(S^4, kS^2\right)$, and let X_0 denote a midsection.

Construct an algorithm that computes from the midsection X_0 a presentation of $\pi_2 X$ as a $\mathbb{Z}\pi_1 X$ -module that is as efficient and as easy to compute as the Wirtiger presentation.

Objective 2

Do the same for the k-invariant:

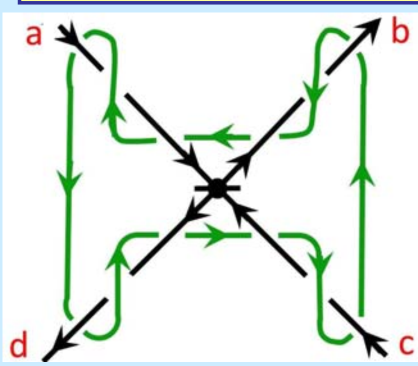
$$kx \in H^3\left(\pi_1 X; \pi_2 X\right)$$

Conjecture

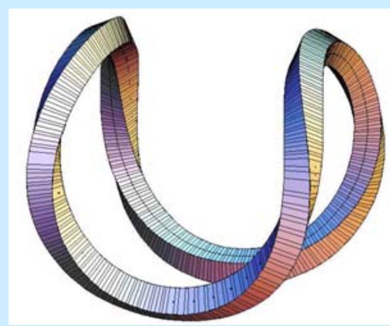
The baseball generators form a complete set of generators for $\pi_2 X$ as a $\mathbb{Z}\pi_1 X$ module.

Baseball Seam

Baseball Generator β



Baseball Seam



Theorem (Lomonaco): Let X be the exterior of a 2-knot $\left(S^4, kS^2\right)$, and let X_0 denote a midsection. Moreover, let

$$H = Ker\left(\pi_1 X_0 \to \pi_1 X_U \times \pi_1 X_D\right)$$

Then

$$\pi_2 X \cong \pi_1 X_0 / \left[H, H\right]$$

Observation: Let baseball curve β is an element of the kernel H which does not lie in the commutator group $\left[H, H\right]$.

R's & I's ←→ Saddle Pt

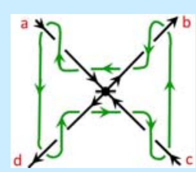
UpSection:

Up-R's: $\hat{h}_{UT} = a\bar{b}, \hat{h}_{UB} = c\bar{d}$

Up-Id's: $\hat{I}_U = h_{UT} \cdot h_{UB}^{b\bar{c}} \cdot \bar{h}$

Mid-Section:

Mid-R: $\hat{h} = a\bar{d}c\bar{b}$

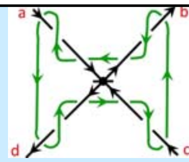


Down-Section:

Down-R's: $\hat{h}_{DL} = a\bar{d}, \hat{h}_{DR} = c\bar{b}$

Down-Id's: $\hat{I}_D = h_{DL} \cdot h_{DR} \cdot \bar{h}$

R's & I's ←→ Saddle Pt



UpSection:

Up-R's: h_{UT} & h_{UB} ←→ 2-cells (Attached)

Up-Id's: I_U ←→ 3-cell (Attached)

Mid-Section:

Mid-R: h ←→ 2-cell

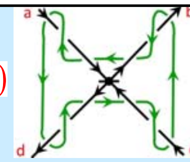
Down-Section:

Down-R's: h_{DL} & h_{DR} ←→ 2-cells (Attached)

Down-Id's: I_D ←→ 3-cell (Attached)

Baseball 2-Cycle

$$\beta = \bar{b}c\bar{d}a \in Ker(\pi_1 X_0 \to \pi_1 X_U \times \pi_1 X_D)$$



The baseball identity is:

$$\hat{I}_B = h_{UT}^{a} \cdot h_{UB}^{\bar{a}} \cdot \bar{h}_{DL}^{\bar{a}} \cdot \bar{h}_{DR}^{\bar{c}}$$

The baseball identity I_B does NOT correspond to a 3-cell in the knot exterior.

Hence, $\hat{I}_B$ corresponds to a non-bounding 2-cycle in the universal covering of the knot exterior.

More to come

There is also an algorithm reading off the relators which is conjectured to be complete. But there is not enough time left to explain it.

This talk based on:

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And also based on:

Fox, R.H., A quick trip through knot theory, in "Topology of 3-Manifolds and Related Topics," ed. by M.K. Fort, Jr., Prentice-Hall, Englewoods Cliffs, New Jersey, (1962), 120-167.

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