1

Quantum Computing:

Lecture 3

Samuel J. Lomonaco, Jr.
Dept. of Computer Science & Electrical Engineering
University of Maryland Baltimore County
Baltimore, MD 21250
Email: Lomonaco@UMBC.EDU
WebPage: http://www.csee.umbc.edu/~lomonaco

Transcribed into PowerPoint by Mark Laczin

2

Lecture 3: Part 1

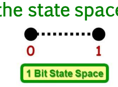
Visualizing Qubits:

The Bloch Sphere

3

Visualizing Qubits

- It's easy to visualize the state space $\{0, 1\}$ of a Shannon bit, i.e.,



- But how do we visualize the state of a qubit?
- So far, we've thought of a single qubit as a vector in a 2-D complex Hilbert space. But this is actually not fully correct!

4

Visualizing Qubits

- Let $|\psi\rangle \in \mathcal{H}$.
- Actually, for all $\lambda \in \mathbb{C} - \{0\}$, $|\psi\rangle$ *and* $\lambda|\psi\rangle$ represent the same physical state, because a global phase λ is not physically detectable. Hence, $\lambda|\psi\rangle$ and $|\psi\rangle$ represent the same state, i.e., $\lambda|\psi\rangle \cong |\psi\rangle$.
- More precisely, the complex line through the origin of $\mathcal{H}$ is actually the state, i.e.,

$$\{\lambda|\psi\rangle \mid \lambda \in \mathbb{C}\}$$

is the state.

5

Visualizing Qubits

- The space of all complex lines through the origin is called the complex projective space and is denoted $\mathbb{C}P^1$.
- We now show how to visualize the qubit state space $\mathbb{C}P^1$ as the surface of a ball, i.e., a 2-sphere.

6

Visualizing Qubits

- Consider an arbitrarily chosen state of a qubit:

$$|\psi\rangle = a|0\rangle + b|1\rangle = \binom{a}{b}$$

- For the time being, assume that the amplitude $a \neq 0$. Then the state is equivalent to

$$\frac{1}{a}|\psi\rangle = |0\rangle + \frac{b}{a}|1\rangle,$$

where $\frac{b}{a} = s + it$ is a complex number.

7

Visualizing Qubits

- In this way, the states of a quantum system correspond to the complex numbers $\mathbb{C}$:

$$\{\lambda|\psi\rangle\} \leftrightarrow \frac{b}{a} \in \mathbb{C}.$$

- If $a = 0$, we say that the state corresponds to a point at infinity ∞ .
- So, the state space of a qubit can be identified with the complex numbers $\mathbb{C}$ plus infinity ∞, i.e., $\mathbb{C} \cup \infty$.

8

Visualizing Qubits: The Complex Plane

$$|0\rangle \leftrightarrow {}^{0}/_{1} = 0$$

$$|1\rangle \leftrightarrow {}^{1}/_{0} = \infty$$

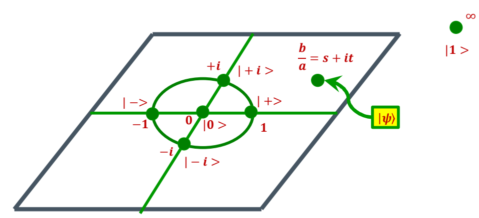
$$|+\rangle = \frac{(|0\rangle + |1\rangle)}{\sqrt{2}} \leftrightarrow +1 \qquad |i\rangle = \frac{(|0\rangle + i|1\rangle)}{\sqrt{2}} \leftrightarrow i$$

$$|-\rangle = \frac{(|0\rangle - |1\rangle)}{\sqrt{2}} \leftrightarrow -1 \qquad |-i\rangle = \frac{(|0\rangle - i|1\rangle)}{\sqrt{2}} \leftrightarrow -i$$

$$|\psi\rangle = a|0\rangle + b|1\rangle \rightarrow {}^{b}/_{a} = s + it$$

9

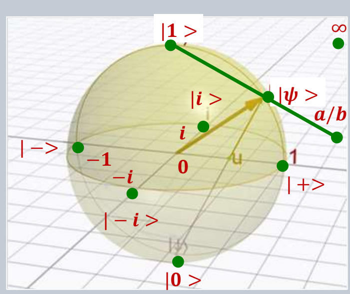
Visualizing Qubits: The Complex Plane



10

VISUALIZING QUBITS: THE BLOCH SPHERE

We now use the reverse stereographic projection to identify $\mathbb{C} \cup \infty$ with the surface of a ball, i.e., the 2-sphere.



11

Visualizing Qubits: The Bloch Sphere

Thus, the state space of a qubit is a sphere, called the Bloch sphere.

12

Lecture 3: Part 2

Multipartite Systems:
Beyond a Single Qubit

13

Definitions

- A multipartite quantum system $\mathcal{Q}$ consists of n qubits

$$\mathcal{Q}_0, \mathcal{Q}_1, \dots, \mathcal{Q}_{n-1},$$

with respective Hilbert spaces

$$\mathcal{H}_0, \mathcal{H}_1, \dots, \mathcal{H}_{n-1},$$

and has as its state space the Hilbert space

$$\mathcal{H} = \otimes_{j=0}^{n-1} \mathcal{H}_j.$$

14

Definitions

- If $\{|0\rangle, |1\rangle\}$ is designated as the standard orthonormal basis of $\mathcal{H}_j$, then these bases induce a standard orthonormal basis

$$|0\rangle, |1\rangle, \dots, |2^{n-1}\rangle$$

of the n qubit state space $\mathcal{H}$, where the above integer labels denote the corresponding n bit binary strings.

15

The State of $|\psi\rangle$

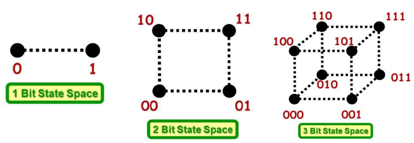
- Thus, the state $|\psi\rangle$ of n qubits is of the form

$$|\psi\rangle = \sum_{j=0}^{2^n-1} z_j|j\rangle = z_0|0\rangle + z_1|1\rangle + \cdots + z_{2^n-1}|2^n - 1\rangle$$

16

Shannon Bits

Note that it is easy to visualize the state space of n Shannon bits.



- This is because the state space of Shannon bits,

$$\{0,1\}^n = \{0,1\} \times \{0,1\} \times \cdots \times \{0,1\},$$

is simply the set of all binary n bit strings.

17

Bits versus Qubits

- For classical bits, please note that the number of states of an n bit multipartite system is $O(2^n)$, but each state can be easily designated by an $O(n)$-bit string.
- For qubits, the state space $\mathcal{H}$ of a multipartite quantum system is much more complex, and more difficult to visualize.

18

Question: How to visualize an n-qubit state?

- Let $|\psi\rangle = \sum_{j=0}^{2^n-1} z_j|j\rangle$ be a state of n qubits.
- Case 1: if $z_0 \neq 0$, then this is the same physical state as

$$|\psi\rangle = |0\rangle + \sum_{j=1}^{2^n-1} \frac{z_j}{z_0}|j\rangle \leftrightarrow \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \dots, \frac{z_{2^n-1}}{z_0}\right) \in \mathbb{C}^{2^n-1}$$

- Case 2: il $z_0 = 0$ and $z_1 \neq 0$, then

$$|\psi\rangle = |1\rangle + \sum_{j=2}^{2^n-1} \frac{z_j}{z_1}|j\rangle \leftrightarrow \left(\frac{z_2}{z_1}, \frac{z_3}{z_1}, \dots, \frac{z_{2^n-1}}{z_1}\right) \in \mathbb{C}^{2^n-2}$$

- Case 3: etc…

19

Question continued…

- It follows that the state space of n qubits can be identified with

$$\mathbb{C}^{2^n-1} \sqcup \mathbb{C}^{2^n-2} \sqcup \cdots \sqcup \mathbb{C},$$

which is a $(2^n - 1)$-dimensional complex space, which in turn is a $2(2^n - 1)$-dimensional Real space. This space is the complex projective space $\mathbb{C}P^{2^n-1}$.

20

Lecture 3: Part 3

Measurement:

Classical and Quantum

21

Measurement: W.r.t the $\{|0\rangle, |1\rangle\}$ basis

- Recall that if a qubit in the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is measured with respect to the standard basis $\{|0\rangle, |1\rangle\}$, then

$$|\psi\rangle \to \boxed{\text{Measurement w.r.t. } \{|0\rangle, |1\rangle\}} \to \begin{cases} |0\rangle, & \wp_0 = |\alpha|^2 \\ |1\rangle, & \wp_1 = |\beta|^2 \end{cases}$$

where we have assumed that $|\psi\rangle$ is of unit length, i.e.,

$$\langle\psi|\psi\rangle = |\alpha|^2 + |\beta|^2 = 1.$$

22

Measurement: W.r.t. other bases, e.g., $\{|+\rangle, |-\rangle\}$

- Now we also note that measurement may be made with respect to any orthonormal basis.
- For example, let us consider the orthnormal basis

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

- If we re-express $|\psi\rangle$ in this basis, we have

$$|\psi\rangle = \frac{(\alpha + \beta)}{2}|+\rangle + \frac{(\alpha - \beta)}{2}|-\rangle$$

- Then,

$$|\psi\rangle \to \boxed{\text{Measurement w.r.t. } \{|+\rangle, |-\rangle\}} \to \begin{cases} |+\rangle, & \wp_+ = \left|\frac{(\alpha+\beta)}{2}\right|^2 \\ |-\rangle, & \wp_- = \left|\frac{(\alpha-\beta)}{2}\right|^2 \end{cases}$$

23

Measurement: W.r.t. the $\{|i\rangle, |-i\rangle\}$ basis

- In like manner, consider the orthnormal basis

$$|i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}} \qquad |-i\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$$

- If we re-express $|\psi\rangle$ in this basis, we have

$$|\psi\rangle = \frac{(\alpha - i\beta)}{2}|i\rangle + \frac{(\alpha + i\beta)}{2}|-i\rangle$$

- Then,

$$|\psi\rangle \to \boxed{\text{Measurement w.r.t. } \{|i\rangle, |-i\rangle\}} \to \begin{cases} |i\rangle, & \wp_i = \left|\frac{(\alpha-i\beta)}{2}\right|^2 \\ |-i\rangle, & \wp_{-i} = \left|\frac{(\alpha+i\beta)}{2}\right|^2 \end{cases}$$

24

Classical versus Quantum Measurement: As coin flips

- The previous three examples of measurements w.r.t. a basis demonstrate very emphatically how a qubit differs from a Shannon bit.
- We can see this with an example. Consider a classical coin, which when flipped (thus observed) produces a heads H or a tails T with probability

$$\begin{cases} H, & p_H \\ T, & p_T \end{cases}$$

where $p_h + p_T = 1$.

25

Measurement: Qubits

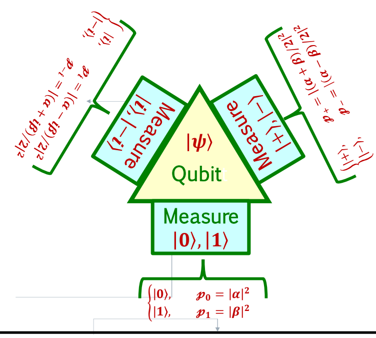
- Let us now compare the classical coin flip with the measurement of a qubit in the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$.
- Then, unlike a classical coin, the qubit behaves as 3 different coins, depending on which of the three bases

$$\{|0\rangle, |1\rangle\}, \quad \{|+\rangle, |-\rangle\}, \quad \{|i\rangle, |-i\rangle\}$$

is chosen, as shown in the following figure:

26

Measurement of a qubits



27

Measurement: more qubits

- In other words, a qubit behaves like many different coins, depending on how it is observed.
- If, in the same vein, we were to continue to pursue this for 2 qubits, we would encounter some amazing non-classical behavior, e.g., the Bell inequalities.

28

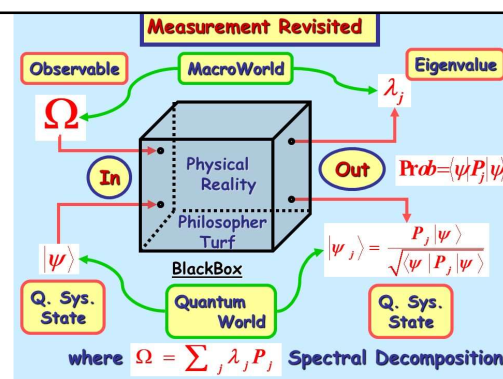
There is much more to Q Measurement

Step 1. Measurement of a qubit with respect to orthonormal basis

Step 2. Measurement of n qubits with respect to orthonormal basis

Step 3. Measurement of with respect to eigen basis of a Hermitian operator (a.k.a., an observable

29



30

Lecture 3: Part 4

Wiring Diagrams:
Drawing Quantum Circuits

31

Wiring Diagrams: Recall

- As previously mentioned, there are two types of quantum computer instruction:

1. Unitary transformations $\boldsymbol{U}$, where $\boldsymbol{U}^{\dagger} = \boldsymbol{U}^{-1}$.
2. Measurements, which are given by Hermitian matrices $\boldsymbol{\Omega}$, where $\boldsymbol{\Omega}^{\dagger} = \boldsymbol{\Omega}$.

- However, we now encounter a problem…

32

A Problem

- As the number $\boldsymbol{n}$ of qubits grows, the size of the matrices $\boldsymbol{U}$ and $\boldsymbol{\Omega}$ grow as $\mathbf{2}^{\boldsymbol{n}} \times \mathbf{2}^{\boldsymbol{n}}$, i.e., *exponentially!*
- Question: How do we program a quantum computer when the size of the instructions grows as $\boldsymbol{O}(\mathbf{2}^{\boldsymbol{n}})$?
- The answer is to represent quantum computer programs as wiring diagrams which grow linearly $\boldsymbol{O}(\boldsymbol{n})$ with $\boldsymbol{n}$.
- In a sense, the language of wiring diagrams is analogous to today's classical computer languages, e.g., C/C++, Python, etc.

33

What is a wiring diagram?

- A wiring diagram consists of lines and gates.
- A solid line ——— corresponds to a qubit, and a double line ═══ corresponds to a classical Shannon bit.
- A solid line labeled with a slash —/ⁿ— denotes $\boldsymbol{n}$ qubits.
- Gates correspond to quantum computer instructions, i.e., to unitary transformations and measurements.

34

Some single qubit gates

- Hadamard: —[H]— $= \left(\frac{1}{\sqrt{2}}\right)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
- Pauli-X: —[X]— $= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- Pauli-Y: —[Y]— $= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

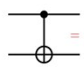
35

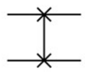
Some single qubit gates

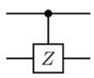
- Pauli-Z: —[Z]— $= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- Phase: —[S]— $= \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$
- $\frac{\pi}{8}$: —[T]— $= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$

36

Some multi-qubit gates pt. 1

› Controlled-NOT:  $= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

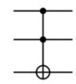
› SWAP:  $= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

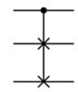
› Controlled-Z: 

$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

37

Some multi-qubit gates

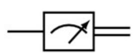
› Toffoli: 

› Fredkin: 

38

Measurement gate

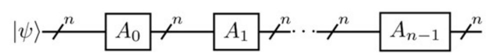
› Measurement is represented by a small meter:



39

Composition

› In general, a wiring diagram of the form



And represents the unitary transformation

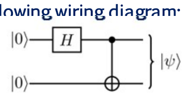
$$A_{n-1}A_{n-2}\cdots A_1A_0.$$

› Note the corresponding order reversal of the product!

40

Example 1

› Consider the following wiring diagram:



› This is equivalent to

$$(CNOT)(H \otimes I)(|0\rangle \otimes |0\rangle)$$

› That is,

$$|\psi\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \left[\left(\frac{1}{\sqrt{2}}\right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

41

Example 1 cont.

› Performing the arithmetic,

$$|\psi\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

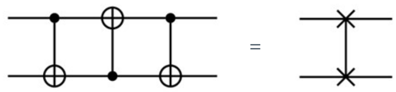
$$= \left(\frac{1}{\sqrt{2}}\right) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \left(\frac{1}{\sqrt{2}}\right) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

› which is known as a *Bell state*.

42

Exercise 1

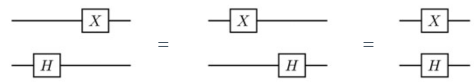
› Show that



43

Exercise 2

› Show that

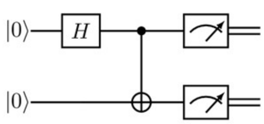


› Note that in general,

$$(A \otimes I)(I \otimes B) = (I \otimes B)(A \otimes I) = (A \otimes B)$$

44

Now consider…



$$= \begin{cases} |00\rangle, \; p_{00} = 1/2 \\ |01\rangle, \; p_{01} = 0 \\ |10\rangle, \; p_{10} = 0 \\ |11\rangle, \; p_{11} = 1/2 \end{cases}$$

45

The Bell Orthonormal Basis

› Consider the Bell orthnormal basis of a 2-qubit state space:

› $|\beta_{00}\rangle = (I \otimes I)|\beta_{00}\rangle = \frac{(|00\rangle + |11\rangle)}{\sqrt{2}}$

› $|\beta_{01}\rangle = (X \otimes I)|\beta_{00}\rangle = \frac{(|10\rangle + |01\rangle)}{\sqrt{2}}$

› $|\beta_{10}\rangle = (Z \otimes I)|\beta_{00}\rangle = \frac{(|00\rangle - |11\rangle)}{\sqrt{2}}$

› $|\beta_{11}\rangle = (ZX \otimes I)|\beta_{00}\rangle = \frac{(|01\rangle - |10\rangle)}{\sqrt{2}}$

› Thus, $|\beta_{ab}\rangle = (Z^a X^b \otimes I)|\beta_{00}\rangle$ where $a, b \in \{0, 1\}$.

46

Note

› We can move from one Bell basis to any other by applying a local unitary transformation to only the left qubit.

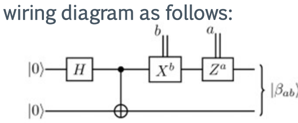
› Later on, this will be useful in quantum communication protocols.

› Exercise 3: Verify the formulas relating the Bell basis elements.

47

Transforming the Bell Basis

We can translate the equation $|\beta_{ab}\rangle = (Z^a X^b \otimes I)|\beta_{00}\rangle$
Into a wiring diagram as follows:



› where gates X^b and Z^a are gates controlled by external classical bits a, b. In other words,

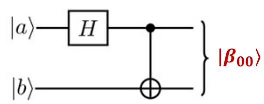
$$X^b = \begin{cases} I, & if\ b = 0 \\ X, & if\ b = 1 \end{cases} \text{ and } Z^a = \begin{cases} I, & if\ a = 0 \\ Z, & if\ a = 1 \end{cases}$$

› Please note that the first two gates from the left produce the Bell state $|\beta_{00}\rangle$.

48

Exercise 4

› Verify that

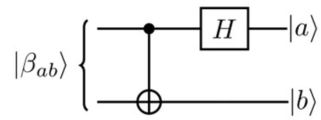


› Thus, the above unitary operations change the standard basis into the Bell basis.

49

Exercise 5

› Prove that

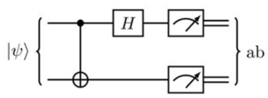


› Thus, the above unitary operation transforms the Bell basis into the standard basis.

50

Bell Basis Measurement

› This circuit makes a Bell basis measurement, i.e., a measurement with respect to the Bell basis:



if $p_{ab} = |\langle \beta_{ab} | \psi \rangle|^2$, where it can be shown that

$$|\psi\rangle = \sum_{a,b=0}^{1} \langle \beta_{ab} | \psi \rangle |\beta_{ab}\rangle$$

51

Lecture 3: Part 5

Quirk: A useful wiring diagram simulator.

https://algassert.com/quirk

52

53