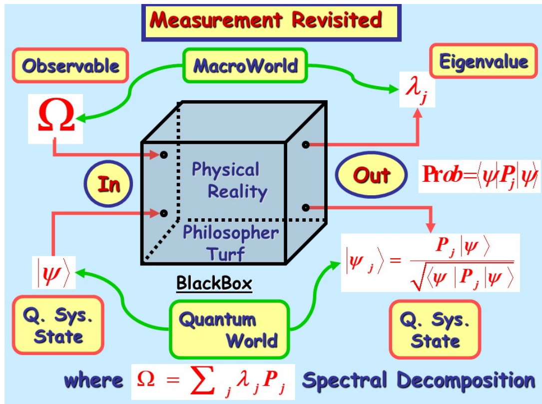
Homework 3 Handout: Illustrative Example



The above graphic summarizes the method used in the calculation below.

Let Q be a two qubit quantum system with state given by the ket |ψ>, where

|ψ> = (|00 > +*i*|01 > - |11 >)/Sqrt[3]

In[51]:=

```
(* Question:  What is the result of measuring  Q  with respect to the observable  Ω,
where  *)
```

$$\Omega = \begin{pmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & i & -1 \\ 1 & -i & 0 & 0 \\ i & -1 & 0 & 0 \end{pmatrix}$$

Out[51]=

```
{{0, 0, 1, -i}, {0, 0, i, -1}, {1, -i, 0, 0}, {i, -1, 0, 0}}
```

In[52]:=

```
(* To save type from repeatingly typing MatrixForm, we define function MF *)
MF[x_] = MatrixForm[x];
```

In[53]:=

```
(* Hence,  *)
```

$$\psi = \left(1/\text{Sqrt}[3]\right)\begin{pmatrix} 1 \\ i \\ 0 \\ -1 \end{pmatrix}; \text{ MF}[\psi]$$

Out[53]//MatrixForm=

$$\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} \\ 0 \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$$

Question: What is the result of measuring Q with respect to the observable Ω, i.e., w.r.t.

In[54]:=

$$\Omega = \begin{pmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & i & -1 \\ 1 & -i & 0 & 0 \\ i & -1 & 0 & 0 \end{pmatrix}; \text{ MF}[\Omega]$$

Out[54]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & i & -1 \\ 1 & -i & 0 & 0 \\ i & -1 & 0 & 0 \end{pmatrix}$$

In[55]:=
```
{eigenval, eigenvec} = Eigensystem[Ω]
```

Out[55]=

$$\left\{\left\{-\sqrt{2}, -\sqrt{2}, \sqrt{2}, \sqrt{2}\right\}, \left\{\left\{\frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 1\right\}, \left\{-\frac{1}{\sqrt{2}}, -\frac{i}{\sqrt{2}}, 1, 0\right\}, \left\{-\frac{i}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 1\right\}, \left\{\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, 1, 0\right\}\right\}\right\}$$

In[56]:=
```
(* λ = List of eigenvalues written as a column vector *)
λ = Transpose[{{eigenval[[3]], eigenval[[4]], eigenval[[1]], eigenval[[2]]}}];
MatrixForm[λ]
(* VPlus = Basis of Eigen Space for eingenvalue  +Sqrt[2] *)
(* VMinus = Basis of Eigen Space for eingenvalue -Sqrt[2] *)
VPlus = {eigenvec[[3]], eigenvec[[4]]}; MF[VPlus]
VMinus = {eigenvec[[1]], eigenvec[[2]]}; MF[VMinus]
```

Out[56]//MatrixForm=

$$\begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ -\sqrt{2} \\ -\sqrt{2} \end{pmatrix}$$

Out[57]//MatrixForm=

$$\begin{pmatrix} -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 1 & 0 \end{pmatrix}$$

Out[58]//MatrixForm=

$$\begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 1 \\ -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 1 & 0 \end{pmatrix}$$

In[59]:=
```
(* Caveate:  Please keep in mind that,
if an eigen space is of dimension greater than 1,
then its orthonormal basis is not
 unique.                                                     *)
```

In[60]:=
```
(* Problem:  Then each of the two bases may not be orthonormal bases. *)
(* So we need to use the GrammSchmidt orthogonalization algorithm to *)
(* transform each basis to an orthogonal basis and then we need to   *)
(* normaize each basis vector in each of the two bases. *)
```

In[61]:=
```
VPlusPerp = Orthogonalize[VPlus]
VMinusPerp = Orthogonalize[VMinus]
```

Out[61]=

$$\left\{\left\{-\frac{i}{2}, -\frac{1}{2}, 0, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{2}, \frac{i}{2}, \frac{1}{\sqrt{2}}, 0\right\}\right\}$$

Out[62]=

$$\left\{\left\{\frac{i}{2}, \frac{1}{2}, 0, \frac{1}{\sqrt{2}}\right\}, \left\{-\frac{1}{2}, -\frac{i}{2}, \frac{1}{\sqrt{2}}, 0\right\}\right\}$$

In[63]:=
```
w1 = Transpose[{VPlusPerp[[1]]}]; MF[w1]
w2 = Transpose[{VPlusPerp[[2]]}]; MF[w2]

w3 = Transpose[{VMinusPerp[[1]]}]; MF[w3]
w4 = Transpose[{VMinusPerp[[2]]}]; MF[w4]
```

Out[63]//MatrixForm=

$$\begin{pmatrix} -\frac{i}{2} \\ -\frac{1}{2} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Out[64]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

Out[65]//MatrixForm=

$$\begin{pmatrix} \frac{i}{2} \\ \frac{1}{2} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Out[66]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{2} \\ -\frac{i}{2} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

In[67]:=
```
(* Check to se if above kets are of unit length *)
w1† . w1;
w2† . w2;
w3† . w3;
w4† . w4;
```

In[71]:=
```
(* Check to see if above kets are orthonal *)
w1† . w2;
w1† . w3;
w1† . w4;
w2† . w3;
w2† . w4;
w3† . w4;
```

In[77]:=
```
(* Computation of the corresponding projections *)
```

In[78]:=
```
P1 = w1 . w1†; MF[P1]
P2 = w2 . w2†; MF[P2]
P3 = w3 . w3†; MF[P3]
P4 = w4 . w4†;
MF[P4]
```

Out[78]//MatrixForm=

$$\begin{pmatrix} \frac{1}{4} & \frac{i}{4} & 0 & -\frac{i}{2\sqrt{2}} \\ -\frac{i}{4} & \frac{1}{4} & 0 & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ \frac{i}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & \frac{1}{2} \end{pmatrix}$$

Out[79]//MatrixForm=

$$\begin{pmatrix} \frac{1}{4} & -\frac{i}{4} & \frac{1}{2\sqrt{2}} & 0 \\ \frac{i}{4} & \frac{1}{4} & \frac{i}{2\sqrt{2}} & 0 \\ \frac{1}{2\sqrt{2}} & -\frac{i}{2\sqrt{2}} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[80]//MatrixForm=

$$\begin{pmatrix} \frac{1}{4} & \frac{i}{4} & 0 & \frac{i}{2\sqrt{2}} \\ -\frac{i}{4} & \frac{1}{4} & 0 & \frac{1}{2\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ -\frac{i}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & \frac{1}{2} \end{pmatrix}$$

Out[81]//MatrixForm=

$$\begin{pmatrix} \frac{1}{4} & -\frac{i}{4} & -\frac{1}{2\sqrt{2}} & 0 \\ \frac{i}{4} & \frac{1}{4} & -\frac{i}{2\sqrt{2}} & 0 \\ -\frac{1}{2\sqrt{2}} & \frac{i}{2\sqrt{2}} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[82]:=
```
(* Verification that above operatours are projectors *)
(P1.P1) == P1;
(P2.P2) == P2;
(P3.P3) == P3;
(P4.P4) == P4;
```

In[86]:=
```
(* Verification that above projectors are orthogonal *)
```

In[87]:=
```
(* Verification that the four projectors for a complete set of projectors *)
P1 + P2 + P3 + P4 == IdentityMatrix[4];
```

In[88]:=
```
(* Verification of the spectral decomposition of Ω *)
Ω == (λ[[1, 1]] * P1) + (λ[[2, 1]] * P2) + (λ[[3, 1]] * P3) + (λ[[4, 1]] * P4)
```

Out[88]= True

In[89]:=
```
(* There are two distinct eigenvalues,
hence two distinct eigen spaces VPlus and VMinus *)
λPlus = +Sqrt[2]
λMinus = -Sqrt[2]
```

Out[89]= $\sqrt{2}$

Out[90]= $-\sqrt{2}$

In[91]:=
```
(* The projectors for the two eingen spaces VPlus and VMinus are respectively: *)
PPlus = P1 + P2; MF[PPlus]
PMinus = P3 + P4; MF[PMinus]
```

Out[91]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2\sqrt{2}} & -\frac{i}{2\sqrt{2}} \\ 0 & \frac{1}{2} & \frac{i}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & -\frac{i}{2\sqrt{2}} & \frac{1}{2} & 0 \\ \frac{i}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & \frac{1}{2} \end{pmatrix}$$

Out[92]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2\sqrt{2}} & \frac{i}{2\sqrt{2}} \\ 0 & \frac{1}{2} & -\frac{i}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{i}{2\sqrt{2}} & \frac{1}{2} & 0 \\ -\frac{i}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & \frac{1}{2} \end{pmatrix}$$

In[93]:=
```
(* Verification that PPlus & PMinus form a complete set of orthogonal projectors *)
PPlus . PPlus == PPlus
PMinus.PMinus == PMinus
(* PPlus.PMinus==MattrixZero[4] *)
PPlus + PMinus == IdentityMatrix[4]
```

Out[93]= True

Out[94]= True

Out[95]= True

In[96]:=
```
(* Verification of the spectral decomposion of Ω *)
Ω == (λPlus * PPlus) + (λMinus * PMinus)
```

Out[96]= True

In[97]:=
```
(* pPlus =
 Probability the measurement of ψ with respect to Ω will produce eigenvalue λPlus *)
(* pMinus = Probability the measurement of ψ with
  respect to Ω will produce eigenvalue λMinus *)
pPlus = Simplify[ (ψ† . PPlus . ψ)[[1, 1]] ]
pMinus = Simplify[ (ψ† . PMinus . ψ)[[1, 1]] ]
```

Out[97]= $\frac{1}{2}$

Out[98]= $\frac{1}{2}$

In[99]:=
```
(* For eigenvalue λPlus= +Sqrt[2], the resulting state is ψPlus, where          *)
ψPlus = Map[Simplify, (PPlus . ψ) / (Sqrt[ (ψ†.PPlus.ψ)[[1, 1]] ]) ]; MF[ψPlus]
```

Out[99]//MatrixForm=

$$\begin{pmatrix} \frac{i+\sqrt{2}}{2\sqrt{3}} \\ \frac{2i+\sqrt{2}}{2\sqrt{6}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix}$$

In[100]:=
```
(* For eigenvalue λMinus= -Sqrt[2], the resulting state is ψMinus, where          *)
ψMinus = Map[Simplify, (PMinus . ψ) / (Sqrt[ (ψ†.PMinus.ψ)[[1, 1]] ]) ]; MF[ψMinus]
```

Out[100]//MatrixForm=

$$\begin{pmatrix} \frac{-i+\sqrt{2}}{2\sqrt{3}} \\ -\frac{-2i+\sqrt{2}}{2\sqrt{6}} \\ -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix}$$