HOW TO FIND A MATRIX REPRESENTATION OF A QUANTUM GATE

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Abstract. In this class handout, we show how to find a matrix representation of a quantum gate.

1. How to find a matrix representation of a linear transformation

Let V be an n-dimensional vector space over a field $\mathbb{F}$ (e.g., $\mathbb{F} = \mathbb{C}$), and let

$$f : V \longrightarrow V$$

be a linear transformation from V to V.

Given a basis $\underline{\beta}$, i.e.,

$$\underline{\beta} = \{\beta_0, \beta_1, \beta_2, \ldots, \beta_{n-1}\} \ ,$$

the linear transformation is completely detemined by the images of all the basis elements under f, i.e., by

$$f(\beta_0), f(\beta_1), f(\beta_2), \ldots, f(\beta_{n-1}) \ .$$

For given any vector $\psi \in V$ can be written as a linear combination of the basis elements, i.e.,

$$\psi = \sum_{j=0}^{n-1} c_j \beta_j \ ,$$

where $c_j \in \mathbb{F}$ for all j. Thus,

$$f(\psi) = f\left(\sum_{j=0}^{n-1} c_j \beta_j\right) = \sum_{j=0}^{n-1} c_j f(\beta_j) \ .$$

We now identify the vector space V with the vector space $\mathbb{F}^n$ of n-tuple column vectors over $\mathbb{F}$, i.e.,

$$\psi = \sum_{j=0}^{n-1} c_j \beta_j = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{pmatrix} \ .$$

The matrix M of f associated with the basis β is given by

$$M = (f(\beta_0), f(\beta_1), f(\beta_2), \ldots, f(\beta_{n-1})) \ ,$$

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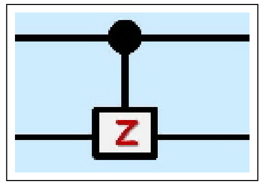
where $f(\beta_j)$ denotes an n-tuple column vector in $\mathbb{F}^n$.

Example 1. *Consider the Controlled-Z gate, i.e.,*



Controlled-Z -gate.

Since

$$\begin{cases} |0\rangle = |00\rangle & \overset{C-Z}{\longmapsto} & |00\rangle = & |0\rangle \\ |1\rangle = |01\rangle & \overset{C-Z}{\longmapsto} & |01\rangle = & |1\rangle \\ |2\rangle = |10\rangle & \overset{C-Z}{\longmapsto} & |10\rangle = & |2\rangle \\ |3\rangle = |11\rangle & \overset{C-Z}{\longmapsto} & -|11\rangle = & -|3\rangle \end{cases}$$

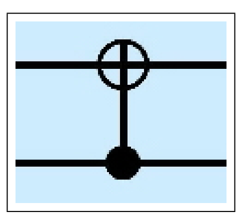
we have

$$C\text{-}Z = \begin{pmatrix} |0\rangle & |1\rangle & |2\rangle & -|3\rangle \end{pmatrix}$$

$$= \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Example 2. *Consider the $\widetilde{Contr}$-Z gate, i.e.,*



Upside Down Controlled-NOT

Since

$$\begin{cases} |0\rangle = |00\rangle & \overset{\widetilde{C\text{-}Z}}{\longmapsto} & |00\rangle = |0\rangle \\ |1\rangle = |01\rangle & \overset{\widetilde{C\text{-}Z}}{\longmapsto} & |11\rangle = |3\rangle \\ |2\rangle = |10\rangle & \overset{\widetilde{C\text{-}Z}}{\longmapsto} & |10\rangle = |2\rangle \\ |3\rangle = |11\rangle & \overset{\widetilde{C\text{-}Z}}{\longmapsto} & |01\rangle = |1\rangle \end{cases},$$

we have

$$\widetilde{C\text{-}Z} = \begin{pmatrix} |0\rangle & |3\rangle & |2\rangle & |1\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$