

TWO EXAMPLES OF PROOF BY MATHEMATICAL INDUCTION.

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Proposition: *Use the principle of mathematical induction to prove that*

$$P(n) : \sum_{j=1}^n j^2 = \frac{n(n+1)(2n+1)}{6},$$

for all integers $n \geq 1$.

Proof (by weak induction):

Basis Step:

$P(n)$ is true for $n = 1$, for:

$$\sum_{j=1}^1 j^2 = 1^2 = 1 = \frac{1(1+1)(2 \cdot 1 + 1)}{6}$$

Inductive Hypothesis:

Assume for a fixed but arbitrary integer $k \geq 1$ that $P(k)$ is true, i.e., that

$$\sum_{j=1}^k j^2 = \frac{k(k+1)(2k+1)}{6}$$

Inductive Step:

[Scratch: We wish to use the Inductive Hypothesis to show that $P(k+1)$ is true, i.e., that

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

We start with the left hand side and transform it using the inductive hypothesis into the right hand side.]

$$\begin{aligned}
\sum_{j=1}^{k+1} j^2 &= \left(\sum_{j=1}^k j^2 \right) + (k+1)^2 && \textbf{Reason: Basic algebra} \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \textbf{Reason: Ind. Hypoth. \& substitution} \\
&= \frac{k(k+1)(2k+1)+6(k+1)^2}{6} && \textbf{Reason: Basic algebra} \\
&= \frac{(k+1)}{6} [k(2k+1) + 6(k+1)] && \textbf{Reason: Basic algebra} \\
&= \frac{(k+1)}{6} [2k^2 + k + 6k + 6] && \textbf{Reason: Basic algebra} \\
&= \frac{(k+1)}{6} (2k^2 + 7k + 6) && \textbf{Reason: Basic algebra} \\
&= \frac{(k+1)}{6} (k+2)(2k+3) && \textbf{Reason: Basic algebra} \\
&= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6} && \textbf{Reason: Basic algebra}
\end{aligned}$$

Thus, we have used the inductive hypothesis to prove that

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

Magic Wand Step:

By the P.M.I., $P(n)$ for all $n \geq 1$, i.e.,

$$\sum_{j=1}^n j^2 = \frac{n(n+1)(2n+1)}{6} \text{ for all } n \geq 1$$

Q.E.D.

Proposition:

Let d_1, d_2, d_3, \dots be the sequence defined by

$$d_j = d_{j-1} \cdot d_{j-2} \text{ for all integers } j \geq 3$$

and

$$d_1 = \frac{9}{10} \quad \text{and} \quad d_2 = \frac{10}{11}$$

Use math induction to prove that

$$P(n) : d_n \leq 1 \text{ for all integers } n \geq 1.$$

Proof (by strong induction):**Basis Step:**

[Scratch: $1 = a < b = 2$]

Both $P(1)$ and $P(2)$ are true, for:

$$\begin{cases} d_1 = \frac{9}{10} \leq 1 & \text{Reason: Definition of } d_1 \\ d_2 = \frac{10}{11} \leq 1 & \text{Reason: Definition of } d_2 \end{cases}$$

Inductive Hypothesis:

Assume for a fixed but arbitrary integer $k \geq 2$ that $P(\ell)$ is true for $1 \leq \ell \leq k$, i.e., that

$$d_\ell \leq 1 \text{ for } 1 \leq \ell \leq k$$

Inductive Step:

[Scratch: We wish to use the Inductive Hypothesis to show that $P(k+1)$ is true, i.e., that $d_{k+1} \leq 1$.]

$$d_{k+1} = d_k \cdot d_{k-1} \quad \text{Reason: Definition of } d_k$$

$$\text{But } d_k \leq 1 \text{ and } d_{k-1} \leq 1 \quad \text{Reason: Ind. Hypoth.}$$

$$\text{Thus, } d_{k+1} \leq 1 \quad \text{Reason: Basic algebra}$$

Magic Wand Step:

Hence, by the P.M.I., $P(n)$ is true for for $n \geq 1$, i.e.,

$$d_n \leq 1 \text{ for } n \geq 1$$

Q.E.D.