Homework 6
Hint

Here is an example problem illustrating how to complete Homework 6.

Hint: Let $\boldsymbol{V}$ be the linear code over $\boldsymbol{GF(2)}$ given by the following generator matrix

$$\boldsymbol{G} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

a) What is the length $\boldsymbol{n}$ of $\boldsymbol{V}$? What the dimension $\boldsymbol{k}$ of $\boldsymbol{V}$? Please explain your answer.

Since the generator matrix has $\mathbf{7}$ *columns,* $\boldsymbol{n = 7}$. *Since the generator matrix is already in echelon canonical form, the dimension of* $\boldsymbol{V}$ *is equal to the number of non-zero rows. Hence,* $\boldsymbol{k = 4}$.

b) Find a parity check matrix $\boldsymbol{H}$ for $\boldsymbol{V}$.

Since the generator matrix $\boldsymbol{G}$ is already in the form $\boldsymbol{G = (I \mid P)}$, we know that the generator matrix $\boldsymbol{H}$ is $\boldsymbol{H = \left(-P^T \mid I\right)}$. Hence,

$$\boldsymbol{H} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

c) Use the parity check matrix $\boldsymbol{H}$ to list all the elements of the dual code $\boldsymbol{V^{\perp}}$.

Since $\boldsymbol{Dim\left(V^{\perp}\right) = n - k = 7 - 4 = 3}$, *it follows that the space of infowords for* $\boldsymbol{V^{\perp}}$ *is the space of all binary 3-tuples. Thus the matrix*

$$W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

is a matrix whose rows are a complete list of the infowords of the code $V^{\perp}$. *Since* H *is the generator matrix of* $V^{\perp}$, *it follows that rows of the matrix*

$$W \bullet H = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

are a complete list of the codewords of the code $V^{\perp}$.

Suggestion: *Use Maple to compute this matrix. See:*

- http://www.cs.umbc.edu/~lomonaco/f06/653/hwk653/hwk06-hint-mapleworksheet.pdf
- http://www.cs.umbc.edu/~lomonaco/f06/653/hwk653/hwk06-hint.mw)

d) Use the results of **c)** to create a weight distribution table, i.e., a table with two columns labeled respectively j and $A_j^{\perp}$.

j	$A_j^{\perp}$
0	1
1	0
2	0
3	0
4	7
5	0
6	0
7	0

e) Use the table created in **d)** to find the weight enumerator polynomial $A^{\perp}(x)$ of the dual code $V^{\perp}$.

$$A^{\perp}(x) = 1 + 7x^4$$

f) Use the MacWilliams identity to compute the weight enumerator polynomial $A(x)$ of the original linear code V.

MacWilliams identity for the weight enumerator $A(x)$ *of* V *in terms of the weight enumerator* $A^{\perp}(x)$ $V^{\perp}$ *is*

$$2^{n-k} A(x) = (1+x)^n A^{\perp}\left(\frac{1-x}{1+x}\right)$$

Thus,

$$A(x) = 2^{-3}(1+x)^7 A^{\perp}\left(\frac{1-x}{1+x}\right) = \frac{1}{8}(1+x)^7\left(1+7\left(\frac{1-x}{1+x}\right)^4\right)$$

$$= \frac{1}{8}(1+x)^7 + \frac{7}{8}(1+x)^3(1-x)^4 = 1 + 7x^3 + 7x^4 + x^7$$

Suggestion: *Use Maple to compute polynomial. See:*

- http://www.cs.umbc.edu/~lomonaco/f06/653/hwk653/hwk06-hint-mapleworksheet.pdf
- http://www.cs.umbc.edu/~lomonaco/f06/653/hwk653/hwk06-hint.mw)

g) Use the weight enumerator polynomial $A(x)$ to find the minimum distance of V.

Since the weight enumerator $A(x)$ *of* V *is* $A(x) = 1 + 7x^3 + 7x^4 + x^7$*, we know that the code* V *consists of* $\mathbf{16}$ *vectors,* $\mathbf{1}$ *of weight* $\mathbf{0}$*,* $\mathbf{7}$ *of weight* $\mathbf{3}$*,* $\mathbf{7}$ *of weight* $\mathbf{4}$*, and* $\mathbf{1}$ *of weight* $\mathbf{7}$*. Thus, the minimum non-zero weight* d *of* V *is* $d = 3$*.*