Homework 1.5
CMSC 643
Quantum Computation
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1 Example Problem

Let $\mathcal{Q}$ be a quantum system with state given by the density operator:

$$\rho = \begin{pmatrix} \frac{1}{4} & -\frac{i}{12} & \frac{1}{12} & \frac{i}{12} \\ \frac{i}{12} & \frac{1}{4} & -\frac{i}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{i}{12} & \frac{1}{4} & -\frac{i}{12} \\ -\frac{i}{12} & \frac{1}{12} & \frac{i}{12} & \frac{1}{4} \end{pmatrix}$$

What is the result of measuring $\mathcal{Q}$ with respect to the observable:

$$\mathcal{O} = \begin{pmatrix} 0 & -1 & -i & 0 \\ -1 & 0 & 0 & i \\ i & 0 & 0 & 1 \\ 0 & -i & 1 & 0 \end{pmatrix}$$

Answer to Example Problem

Remark 1 *Please note that, since* $Tr\left(\rho^2\right) = \frac{1}{3} < 1$, *it follows that* ρ *is a mixed ensemble.*

The eigenkets and corresponding eigenvalues of $\mathcal{O}$ are:

Eigenvalue	Orthonormal Eigenkets(s)	Projection Operator
$a_1 = 2$	$\|w_1\rangle = \frac{1}{2}(-1, 1, -i, -i)^T$	$P_1 = \|w_1\rangle\langle w_1\| = \frac{1}{4}\begin{pmatrix} 1 & -1 & -i & -i \\ -1 & 1 & i & i \\ i & -i & 1 & 1 \\ i & -i & 1 & 1 \end{pmatrix}$
$a_2 = -2$	$\|w_2\rangle = \frac{1}{2}(1, 1, -i, i)^T$	$P_2 = \|w_2\rangle\langle w_2\| = \frac{1}{4}\begin{pmatrix} 1 & 1 & i & -i \\ 1 & 1 & i & -i \\ -i & -i & 1 & -1 \\ i & i & -1 & 1 \end{pmatrix}$
$a_3 = 0$	$\|w_3\rangle = \frac{1}{2}\left(0, \sqrt{2}, i\sqrt{2}, 0\right)^T$ $\|w_4\rangle = \frac{1}{2}\left(i\sqrt{2}, 0, 0, \sqrt{2}\right)^T$	$P_3 = \|w_3\rangle\langle w_3\| + \|w_4\rangle\langle w_4\| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & -i & 0 \\ 0 & i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix}$

Remark 2 *We have once again made sure that the eigenkets of the observable $\mathcal{O}$ form an orthonormal basis.*

Hence,

$$\mathcal{O} = a_1 P_1 + a_2 P_2 + a_3 P_3$$

is the spectral decomposition of the observable $\mathcal{O}$.

Remark 3 *As a check, we can verify that*

$$P_1 + P_2 + P_3 = I$$

is the identity operator, and also that $P_j^2 = P_j$ for all j.

Hence, if ρ is measured with respect to the observable $\mathcal{O}$, we obtain:

Probability	Eigenvalue	Resulting State
$p_1 = Tr\left(P_1 \rho\right)$	2	$\rho_1 = \frac{P_1 \rho P_1}{Tr(P_1 \rho)}$
$p_2 = Tr\left(P_2 \rho\right)$	-2	$\rho_2 = \frac{P_2 \rho P_2}{Tr(P_2 \rho)}$
$p_3 = Tr\left(P_3 \rho\right)$	0	$\rho_3 = \frac{P_3 \rho P_3}{Tr(P_3 \rho)}$

,

which when computed is found to be:

Probability	Eigenvalue	Resulting State
$\frac{1}{6}$	2	$\frac{1}{4}\begin{pmatrix} 1 & -1 & -i & -i \\ -1 & 1 & i & i \\ i & -i & 1 & 1 \\ i & -i & 1 & 1 \end{pmatrix}$
$\frac{1}{6}$	-2	$\frac{1}{4}\begin{pmatrix} 1 & 1 & i & -i \\ 1 & 1 & i & -i \\ -i & -i & 1 & -1 \\ i & i & -1 & 1 \end{pmatrix}$
$\frac{2}{3}$	0	$\frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & -i & 0 \\ 0 & i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix}$

Remark 4 *Since $\rho_1 = P_1$ and $\rho_2 = P_2$, it follows that $\rho_1^2 = \rho_1$ and $\rho_2^2 = \rho_2$. Hence, ρ_1 and ρ_2 are pure ensembles. On the other hand, since $\rho_3 = \frac{1}{2}P_3$, it follows that $\rho_3^2 = \frac{1}{2}\rho_3 \neq \rho_3$. Thus, ρ_3 is a mixed ensemble.*

2 Problem 1.5

Let $\mathcal{Q}$ be a quantum system with state given by the density operator:

$$\rho = \begin{pmatrix} \frac{1}{4} & -\frac{i}{12} & \frac{1}{12} & \frac{i}{12} \\ \frac{i}{12} & \frac{1}{4} & -\frac{i}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{i}{12} & \frac{1}{4} & -\frac{i}{12} \\ -\frac{i}{12} & \frac{1}{12} & \frac{i}{12} & \frac{1}{4} \end{pmatrix}$$

What is the result of measuring $\mathcal{Q}$ with respect to the following observables:

a)

$$\mathcal{O} = \begin{pmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & i & -1 \\ 1 & -i & 0 & 0 \\ i & -1 & 0 & 0 \end{pmatrix}$$

b)

$$\mathcal{O} = \begin{pmatrix} 2 & 0 & 0 & i \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -i & 0 & 0 & 2 \end{pmatrix}$$

c)

$$\mathcal{O} = \frac{1}{2} \begin{pmatrix} 5 & 0 & 0 & 3i \\ 0 & 5 & i & 0 \\ 0 & -i & 5 & 0 \\ -3i & 0 & 0 & 5 \end{pmatrix}$$