CLASS NOTES ON MATHEMATICAL INDUCTION

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Definition 1 (Weak Principle of Mathematical Induction (WPMI)). *Let S be a subset of the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. Then*

$$\left.\begin{array}{l} 0 \in S \\ \\ n \in S \longrightarrow n+1 \in S \end{array}\right\} \Longrightarrow S = \mathbb{N}$$

Definition 2 (Strong Principle of Mathematical Inductin (SPMI)). *Let S be a subset of the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. Then*

$$\left.\begin{array}{l} 0 \in S \\ \\ 0, 1, 2, \ldots, n-1, n \in S \longrightarrow n+1 \in S \end{array}\right\} \Longrightarrow S = \mathbb{N}$$

Theorem 1.

$$WPMI \Longleftrightarrow SPMI$$

These two principles can be trivially extended to the following:

Proposition 1 (Extended WPMI). *Let S be a a subset of the set $\mathbb{Z}_{\geq n_0} = \{n \in \mathbb{Z} : n \geq n_0\}$, where $\mathbb{Z}$ denotes the set of integers. Then*

$$\left.\begin{array}{l} n_0 \in S \\ \\ n \in S \longrightarrow n+1 \in S \end{array}\right\} \Longrightarrow S = \mathbb{Z}_{\geq n_0}$$

Proposition 2 (Extended SPMI). *Let S be a a subset of the set $\mathbb{Z}_{\geq n_0} = \{n \in \mathbb{Z} : n \geq n_0\}$, where $\mathbb{Z}$ denotes the set of integers. Then*

$$\left.\begin{array}{l} n_0 \in S \\ \\ n_0, n_0+1, \ldots, n-1, n \in S \longrightarrow n+1 \in S \end{array}\right\} \Longrightarrow S = \mathbb{Z}_{\geq n_0}$$

We will use the PMI to prove that a predicate $P(n)$ is true for all integers $n \geq n_0$ by showing that the set

$$\{n \in Z : (n \geq n_0) \wedge P(n)\}$$

is the same as $\mathbb{Z}_{\geq n_0}$.

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For the WPMI, we will use the following template:

Template for Using the WPMI

Basis Step *$P(n_0)$ is true for the following reason:*

Inductive Hypothesis Step *Assume that for a fixed but arbitrary integer $k \geq n_0$ that $P(k)$ is true.*

Inductive Step *$P(k+1)$ is true for the following reason:*
Use the Inductive Hypothesis in this step.

Magic Wand Step *By the PMI, $P(n)$ is true for all integers $n \geq n_0$.*

Q.E.D.

For the SPMI, we will use the following template:

Template for Using the SPMI

Basis Step *$P(n_0)$ is true for the following reason:*

Inductive Hypothesis Step *Assume that for a fixed but arbitrary integer $k \geq n_0$ that $P(\ell)$ is true for $n_0 \leq \ell \leq k$.*

Inductive Step *$P(k+1)$ is true for the following reason:*
Use the Inductive Hypothesis in this step.

Magic Wand Step *By the PMI, $P(n)$ is true for all integers $n \geq n_0$.*

Q.E.D.

As an example we prove the following:

Proposition 3.

$$\sum_{j=1}^{n} j = \frac{n\,(n+1)}{2}$$

for all integers $n \geq 1$.

Proof. Let $P(n)$ be the statement that $\sum_{j=1}^{n} j = n(n+1)/2$.

Basis Step $P(1)$ is true, for:

$$\sum_{j=1}^{1} j = 1 = \frac{1(1+1)}{2}$$

Inductive Hypothesis Step *Assume that for a fixed but arbitrary integer* $k \geq 1$ *that* $P(k)$ *is true, i.e., that*

$$\sum_{j=1}^{k} j = \frac{k(k+1)}{2}$$

Inductive Step $P(k+1)$ *is true for the following reason:*

$$\sum_{j=1}^{k+1} j = \left(\sum_{j=1}^{k} j\right) + (k+1) \qquad \text{by basic algebra}$$

$$= \frac{k(k+1)}{2} + (k+1) \qquad \text{by the I.H.}$$

$$= \frac{k(k+1) + 2(k+1)}{2} \qquad \text{by basic algebra}$$

$$= \frac{(k+1)(k+2)}{2} \qquad \text{by basic algebra}$$

$$= \frac{(k+1)[(k+1)+1]}{2} \qquad \text{by basic algebra}$$

Hence, we have shown that

$$\sum_{j=1}^{k+1} j = \frac{(k+1)[(k+1)+1]}{2} \quad ,$$

i.e., we have shown that $P(k+1)$ is true.

Magic Wand Step *By the PMI,* $P(n)$ *is true for all integers* $n \geq 1$, *i.e.*

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2} \ ,$$

is true for all integers $n \geq 1$.

Q.E.D. □