Answer to Exercise 32 on Page 109

Let f be a function from a set A to a set B, and let S and T be subsets of A.

Answer to part a)

Proposition 1

$$f\left(S\cup T\right)=f\left(S\right)\cup f\left(T\right)$$

Proof.

$y\in f\left(S\cup T\right)$

$\Longleftrightarrow \exists x\ \ [(x\in S\cup T)\wedge(y=f(x))]$ Reason: Def of $f\left(S\cup T\right)$

$\Longleftrightarrow \exists x\ \ \{[(x\in S)\vee(x\in T)]\wedge(y=f(x))\}$ Reason: Def of "$\cup$"

$\Longleftrightarrow \exists x\ \ \{[(x\in S)\wedge(y=f(x))]\vee[(x\in T)\wedge(y=f(x))]\}$ Reason: Distributive law

$\Longleftrightarrow (\exists x\ \ [(x\in S)\wedge(y=f(x))])\vee(\exists x\ \ [(x\in T)\wedge(y=f(x))])$ Reason: See Note 1 below

$\Longleftrightarrow [y\in f\left(S\right)]\vee[y\in f\left(T\right)]$ Reason: Defs $f\left(S\right)$ & $f(T)$

$\Longleftrightarrow y\in f\left(S\right)\cup f\left(T\right)$ Reason: Def of "$\cup$"

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Note 1. Unfortunately, our textbook fails to state this theorem. See theorem 26 formula 88 on pages 127-128 of "Mathematical Logic" by Stephen Kleene.

Answer to part b)

Proposition 2

$$f\,(S \cap T) \subseteq f\,(S) \cap f\,(T)$$

Proof.

$y \in f\,(S \cap T)$

$\Longleftrightarrow \exists x\ \ [(x \in S \cap T) \wedge (y = f(x))]$ Reason: Def of $f\,(S \cap T)$

$\Longleftrightarrow \exists x\ \ \{[(x \in S) \wedge (x \in T)] \wedge (y = f(x))\}$ Reason: Def of " $\cap$ "

$\Longleftrightarrow \exists x\ \ \{[(x \in S) \wedge (x \in T)] \wedge [(y = f(x))] \wedge (y = f(x))\}$ Reason: Idempotent law

$\Longleftrightarrow \exists x\ \ \{[(x \in S) \wedge (y = f(x))] \wedge [(x \in T) \wedge (y = f(x))]\}$ Reason: Assoc. & Comm.

$\Longrightarrow \{\exists x\ \ [(x \in S) \wedge (y = f(x))]\} \wedge \{\exists x\,[(x \in T) \wedge (y = f(x))]\}$ Reason: See Note 2

$\Longleftrightarrow (y \in f\,(S)) \wedge (y \in f\,(T))$ Reason: Defs $f\,(S)$ & $f(T)$

$\Longleftrightarrow y \in f\,(S) \cap f\,(T)$ Reason: Def of " $\cap$ "

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Note 2. Unfortunately, our textbook fails to state this theorem. See theorem 26 formula 93 on pages 127-128 of "Mathematical Logic" by Stephen Kleene.