Quantum Computing

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DARPA

Overview
Four Talks

Elementary

- ✓ A Rosetta Stone for Quantum Computation
- ✓ Three Quantum Algorithms
- ✓ Quantum Hidden Subgroup Algorithms
- An Entangled Tale of Quantum Entanglement

Advanced

Lecture 4

An Entangled Tale of Quantum Entanglement

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DARPA

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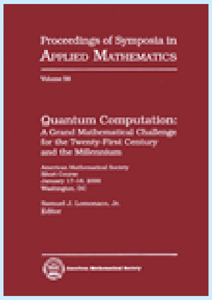
Talk based on the work of many people

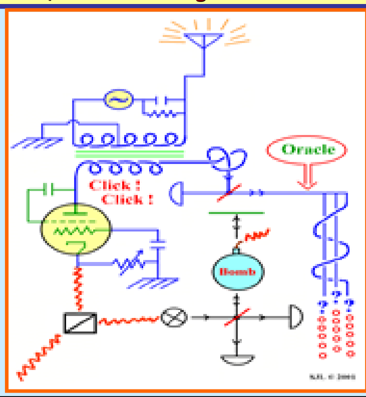
Bennet
Brassard
Brylinski
Nielsen
Horodecki's
Peres
Jonathan
Plenio
Jozsa
Popescu
Linden
Schumacher
Lomonaco
Sudberry
Meyer
Terhal
Wallach
& Many Others

Lomonaco, Samuel J., Jr., An Entangled Tale of Quantum Entanglement, in AMS PSAPM/58, (2002), pages 305 – 349.

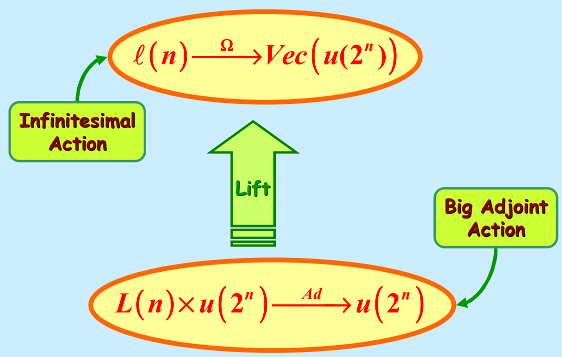


Preamble

Physics Quantum Entanglement Laboratory

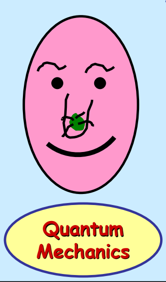


Quantum Entanglement is …

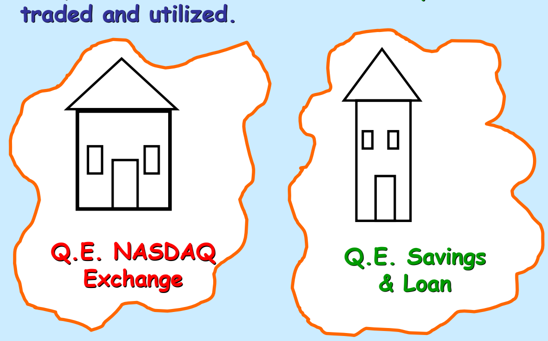


Over the 20-th century, the scientific community's view of Q.E. has dramatically changed.

- Initially, Q.E. was viewed as an unnecessary and unwanted ugly Wart on Quantum Mechanics.
  - EPR tried to surgically remove it.
  - Bell and Aspect showed that surgery can not be performed w/o destroying the very life of physical reality.



Today, Q.E. is viewed as a useful resource in Q.M. It is viewed as a commodity to be traded and utilized.



Quantum Entanglement & Quantum Computation

- Q.E. appears to be an important resource for quantum computation
- Many claim that it is Q.E. that somehow enables us to harness the vast parallelism of Quantum Mechanics.

What is Q.E. ?

??? Questions ???

- How do we { • Measure • Quantify • Classify } Q.E. ?
  - When is the Q.E. of two quantum systems the same? Different?
  - When is the Q.E. of one quantum systems greater than another ?
- Answers to the above questions are expected to have a profound impact on the development of Quantum Computation
- Finding answers to these questions is { • Challenging • Intriguing • Very Habit Forming }

Overview

$L(n) = \bigotimes_{1}^{n} SU(2)$	Local Unitary Group
$\ell(n) = \bigoplus_{1}^{n} su(2)$	Lie algebra of $L(n)$
$U(2^n)$	Unitary Group
$u(2^n)$	Lie algebra of $U(2^n)$

Density Ops "Live" Here

Big Adjoint action of L(n) on u(2^n)

$$L(n) \times u(2^n) \xrightarrow{Ad} u(2^n)$$

The problem of Q.E. "Lives" Here

Overview (Cont.)

$$L(n) \times u(2^n) \xrightarrow{Ad} u(2^n)$$

- We study Q.E. by lifting the above action to the induced infinitesimal action

$$\ell(n) \xrightarrow{\Omega} Vec(u(2^n))$$

- We use the induced infinitesimal action to quantify and classify Q.E. by constructing a complete set of Q.E. invariants.

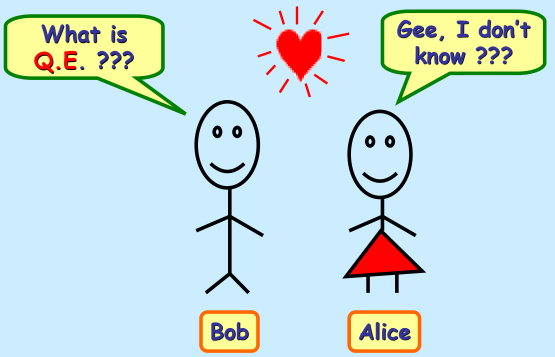
Chapter 1

A Story of Two Qubits

or

How Alice & Bob Learn to Live with Q.E. & Love it

Bob Pops the Big Question



A Story of Two Qubits

Our saga continues

- Alice & Bob visit the local Q.E. wholesale outlet
- They find on one of the shelves a box labeled as follows:

Q.E., Inc.

Two Qubit Quantum System

$\mathcal{Q}_{AB}$

Consisting of Qubits

$\mathcal{Q}_A$ and $\mathcal{Q}_B$

The content label required by federal law reads

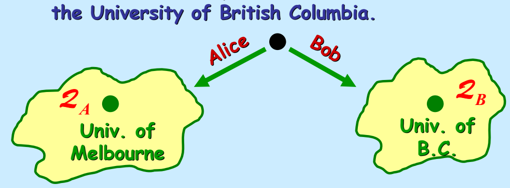
Q Sys	Hilb. Space	State ρ	Unitary Transf.	State Space
$\mathcal{Q}_{AB}$	$\mathcal{H}_{AB}$	ρ_{AB}	$U(2^2)_{AB}$	$u(2^2)_{AB}$
$\mathcal{Q}_A$	$\mathcal{H}_A$	ρ_A	$U(2)_A$	$u(2)_A$
$\mathcal{Q}_B$	$\mathcal{H}_B$	ρ_B	$U(2)_B$	$u(2)_B$

$$\rho_{AB} = \begin{pmatrix} 1/2 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1/2 & 0 & 0 & 1/2 \end{pmatrix}$$

(*) Not legally responsible for the effects of decoherence.

A Story of Two Qubits

- Alice & Bob purchase the two qubit system $\mathcal{Q}_{AB}$.
- Outside the store, they open the box. Alice takes qubit $\mathcal{Q}_A$. Bob takes qubit $\mathcal{Q}_B$.
- They then separate with their respective qubits. Alice flies to the University of Melbourne. Bob flies to Vancouver, British Columbia to attend the University of British Columbia.



A Story of Two Qubits

- After they arrive, Alice has second thoughts about their purchase. She phones Bob, and rattles off in rapid succession the following two questions:
  - Did we get our money's worth of Q.E. ?
  - How much Q.E. did we actually purchase ?

? ? ?

- Bob immediately hangs up, and phones the Q.E. Consumer Protection agency, which refers him to the National Institute of Q.E. Standards & Technology (NIQEST) in Gaithersburg, MD.

- After a long distance ($$$) phone conversation, NIQEST agrees to send Alive & Bob their ***STANDARDS Q.E. KIT***.
- The NIQEST rep. Takes a ***STANDARD*** Q.E. two qubit quantum system $\mathcal{Q}'_{A'B'}$ off the self, places qubit $\mathcal{Q}'_{A'}$ together with a ***STANDARDS MANUAL*** in a Box A. He/She also takes qubit $\mathcal{Q}'_{B'}$ together with a ***STANDARDS MANUAL*** in Box B. He/She then sends the two boxes by overnight mail ($$$) respectively to Alice and Bob.
- After receiving their two boxes, Alice and Bob open them, take out their respective qubits and read the enclosed manuals.

The *STANDARDS MANUAL* reads as follows:

Q.E. Yardstick 1. $\mathcal{Q}_{AB}$ and $\mathcal{Q}'_{A'B'}$ possess the same Q.E. if it is possible for Alice and Bob to use their own local reversible operations (either individually or collectively) to transform $\mathcal{Q}_{AB}$ and $\mathcal{Q}'_{A'B'}$ into one another. If this is possible, then $\mathcal{Q}_{AB}$ and $\mathcal{Q}'_{A'B'}$ are of the same entanglement type, written

$$\mathcal{Q}_{AB} \underset{loc}{\sim} \mathcal{Q}'_{A'B'}$$

The *STANDARDS MANUAL* reads as follows:

Q.E. Yardstick 2. $\mathcal{Q}_{AB}$ possesses more Q.E. than $\mathcal{Q}'_{A'B'}$ if it is possible for Alice and Bob (either individually or collectively) to apply their own reversible and irreversible local operations to their local qubits to transform $\mathcal{Q}_{AB}$ into $\mathcal{Q}'_{A'B'}$. In this case, we write

$$\mathcal{Q}_{AB} \underset{loc}{\geq} \mathcal{Q}'_{A'B'}$$

CAVEAT: Q.E. may be irrevocably lost if Q.E. Yardstick 2 is applied.

Summary

- *Question*. What type of Q.E. do Alice and Bob collectively possess ?
- *Question*. Is the Q.E. of $\mathcal{Q}_{AB}$ the same as the Q.E. of $\mathcal{Q}'_{A'B'}$?
- *Question*. Is the Q.E. of $\mathcal{Q}_{AB}$ greater than the Q.E. of $\mathcal{Q}'_{A'B'}$?

Chapter 2

Definition of the Problem of Quantum Entanglement

Back to Alice & Bob

- Same story for
  - Alice, Bob, & Cathy and 3 qubits
  - n people and n qubits

- Let $\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n$ be n qubits, and let $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$ denote their respective Hilbert spaces
- Let $\mathcal{Q} = \{\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n\}$ be the global quant. sys. with Hilbert Space $\mathcal{H} = \bigotimes_{k=1}^{n} \mathcal{H}_k$

Fundamental Problem of Q.E. (FPQE). Let ρ and ρ' be density operators representing two different states of $\mathcal{Q}$. Is it possible to move $\mathcal{Q}$ from state ρ to state ρ' by applying only local moves ?

Question. But what is a local move ?

Local Moves ?

1) (Standard) local unitary transformations

$$\bigotimes_{k=1}^{n} U_k \in \bigotimes_{k=1}^{n} U(\mathcal{H}_k)$$

For example, $U_A \otimes I, I \otimes U_B, U_A \otimes U_B$

2) Measurement of (Standard) local observables

$$\bigotimes_{k=1}^{n} O_k \in \bigotimes_{k=1}^{n} Observables(\mathcal{H}_k)$$

For example,

$$O_A \otimes I, I \otimes O_B, O_A \otimes O_B$$

Local Moves ?

3) Extended local unitary transformations

$$\bigotimes_{k=1}^{n} U\left(\mathcal{H}_k \otimes \widetilde{\mathcal{H}}_k\right)$$

where, $\mathcal{H}_1, \widetilde{\mathcal{H}}_1, \mathcal{H}_2, \widetilde{\mathcal{H}}_2, \cdots, \mathcal{H}_n, \widetilde{\mathcal{H}}_n$ are distinct Hilbert spaces

4) Measurement of Extended local observables

$$\bigotimes_{k=1}^{n} Observables\left(\mathcal{H}_k \otimes \widetilde{\mathcal{H}}_k\right)$$

where, $\mathcal{H}_1, \widetilde{\mathcal{H}}_1, \mathcal{H}_2, \widetilde{\mathcal{H}}_2, \cdots, \mathcal{H}_n, \widetilde{\mathcal{H}}_n$ are distinct Hilbert spaces

The Group *L(n)* of Local Unitary Transformations

Definition. The group of local unitary transformations *L(n)* is the subgroup of *U(2ⁿ)* defined by

$$L(n) = \bigotimes_{1}^{n} SU(2)$$

Restricted FPQE (RFPQE): Given two density operators $i\rho$ and $i\rho'$ in the Lie algebra $u(2^n)$, does there exist a local move $U \in L(n)$ s.t.

$$Ad_U(i\rho) = U(i\rho)U^{-1} = i\rho' \quad ?$$

Convention. From this point on, we consider only the RFPQE. Thus, for the rest of the talk

Local Moves = *L(n)*

Terminology

Definition. Two elements $i\rho$ and $i\rho'$ in $u(2^n)$ are said to be locally equivalent, written

$$i\rho \underset{loc}{\sim} i\rho'$$

provided there exists a $U \in L(n)$ such that

$$i\rho' = Ad_U(i\rho) = U(i\rho)U^{-1}$$

The equivalence class

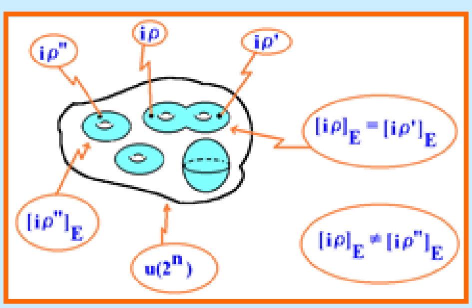
$$[i\rho]_E = \left\{ i\rho' : i\rho' \underset{loc}{\sim} i\rho \right\}$$

is called an entanglement class (or orbit). Finally, let

$$u(2^n)/L(n)$$

denote the set of entanglement classes.

Set of Entanglement Classes $u(2^n)/L(n)$



But What Is Q.E. ?

Lest we forget, our objective is to measure, quantify, classify Q.E.

We will achieve the following two objectives:

Objective 1. Find the dimension of each entanglement class. Given $i\rho$, find $Dim\left([i\rho]_E\right)$

Objective 2. Find a complete set of invariants which classify all entanglement classes. In other words, find a finite set $\{ f_1, f_2, \ldots, f_K \}$ of real valued functions on $u(2^n)$ which distinguish all entanglement classes, i.e.,

$$i\rho \underset{loc}{\sim} i\rho' \Leftrightarrow f_k(i\rho) = f_k(i\rho') \;\; \forall k$$

Chapter 3

Applications of Lie Group

Perspective

Physics	Math
Hilbert Space $\mathcal{H}$, $Dim(\mathcal{H}) = N$, Unitary Group / Lie Group $U(N)$	
Observables: $\mathcal{O}$ Density Ops: ρ $N \times N$ Hermitian Ops $A^\dagger = \overline{A}^T = A$	Observables: $i\mathcal{O}$ Density Ops: $i\rho$ $N \times N$ Skew Hermitian Ops $\in u(N)$ $(iA)^\dagger = \overline{(iA)}^T = -iA$ where $u(N) =$ Lie algebra of $U(N)$
Dynamics via $U \in U(N)$ $\lvert\psi\rangle \mapsto U\lvert\psi\rangle$ $\rho \mapsto U\rho U^\dagger$	Dynamics via $U \in U(N)$ $\lvert\psi\rangle \mapsto U\lvert\psi\rangle$ $i\rho \mapsto Ad_U(i\rho)$ where $Ad_U(i\rho) = U(i\rho)U^{-1}$ is the Big adjoint rep.

Pauli Spin Matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_{31} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We also denote the **2 x 2** identity matrix by

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Basis for the Lie Algebra *u(2)*

- A vector space basis for ***u(2)*** is

$$\left\{ \xi_0 = -\frac{1}{2} i\sigma_0, \xi_1 = -\frac{1}{2} i\sigma_1, \xi_2 = -\frac{1}{2} i\sigma_2, \xi_3 = -\frac{1}{2} i\sigma_3 \right\}$$

- If $i\rho \in u(2)$, then

$$\begin{aligned} i\rho &= x_0\xi_0 + x_1\xi_1 + x_2\xi_2 + x_3\xi_3 \\ &= x_0\xi_0 + \vec{x} \bullet \vec{\xi} \end{aligned}$$

where

$$\begin{cases} x_0 = -1 & \text{if } i\rho \text{ is a density op} \\ \vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \\ \vec{\xi} = (\xi_1, \xi_2, \xi_3) \in u(2) \times u(2) \times u(2) \end{cases}$$

Coordinate Chart for *u(2)*

$$\begin{aligned} \pi : u(2) &\to \mathbb{R}^4 \\ i\rho &\mapsto (x_0, x_1, x_2, x_3) \end{aligned}$$

Basis for the Lie Algebra *u(2²)*

- A basis for ***u(2²)*** is

$$\left\{ \xi_{00} = -\frac{1}{2} i\sigma_0 \otimes \sigma_0, \xi_{01} = -\frac{1}{2} i\sigma_0 \otimes \sigma_1, \ldots, \xi_{33} = -\frac{1}{2} i\sigma_3 \otimes \sigma_3 \right\}$$
$$= \left\{ \xi_{jk} = -\frac{1}{2} i\sigma_j \otimes \sigma_k : j,k = 0,1,2,3 \right\}$$

- $i\rho = \sum_{j,k=0}^{3} x_{jk}\xi_{jk}$

where $x_{00} = -1$ if $i\rho$ is a density operator

Coordinate Chart for $u(2^2)$

$$\pi : u(2^2) \to \mathbb{R}^{16}$$
$$i\rho \mapsto (x_{00}, x_{01}, x_{02}, x_{033}, x_{10}, \ldots x_{33})$$

Basis for the Lie Algebra $u(2^n)$

- A basis for $u(2^n)$ is

$$\left\{ \xi_{j_1 j_2 \cdots j_n} = -\frac{1}{2} i \bigotimes_{\ell=1}^{n} \sigma_{j_\ell} : j_1, j_2, \ldots, j_n = 0,1,2,3 \right\}$$

- $$i\rho = \sum_{j_1, j_2, \ldots, j_n = 0}^{3} x_{j_1 j_2 \cdots j_n} \xi_{j_1 j_2 \cdots j_n}$$

Coordinate Chart for $u(2^n)$

$$\pi : u(2^n) \to \mathbb{R}^{4^n}$$
$$i\rho \mapsto (x_{00..00}, x_{00..01}, \ldots x_{33..33})$$

Overview

$$u(N) = T_I U(N) = Vec_R(U(N)) = Der(C^\infty U(N))$$

$v \in u(N)$ is simultaneously each of the following:

- An $N \times N$ skew Hermitian matrix
- A tangent vector to $U(N)$ at I
- A right invariant vector field on $U(N)$
- A derivation (directional derivative) on $C^\infty(U(N)) = \{ f : U(N) \to \mathbb{R} : f \text{ smooth} \}$

Also Recall

- In $U(2^n)$, there are 4^n independent directions to move in, namely

$$\left\{ \xi_{j_1 j_2 \cdots j_n} = -\frac{1}{2} i \bigotimes_{\ell=1}^{n} \sigma_{j_\ell} : j_1, j_2, \ldots, j_n = 0,1,2,3 \right\}$$

- For example, from $g \in U(2)$ we could move from g in the direction

$$\xi_2 = -\frac{1}{2} i \sigma_2$$

by following the curve

$$\gamma(t) = e^{t\xi_2} g$$

from $t=0$ on.

The Special Unitary Group $SU(N)$

Lie Group

$$SU(N) = \{ U \in U(N) : \det(U) = 1 \}$$

$$su(N) = \{ v \in u(N) : trace(v) = 0 \}$$

Lie Algebra

- A basis for $su(2^n)$ is:

$$\left\{ \xi_{j_1 j_2 \cdots j_n} = -\frac{1}{2} i \bigotimes_{k=1}^{n} \sigma_{j_k} : \begin{array}{l} j_k \in \{0,1,2,3\}\ \forall k \\ \text{Not all } j_k \text{ are zero} \end{array} \right\}$$

- Therefore $Dim(SU(2^n)) = 4^n - 1$

Big Adjoint & little adjoint

- Big Adjoint *Ad* for global dynamics of iρ

$$U(N)\times u(N)\xrightarrow{Ad} u(N)$$
$$(U, i\rho) \mapsto Ad_U(i\rho) = U(i\rho)U^{-1}$$

- little adjoint *ad* for infinitesimal dynamics of iρ

$$u(N)\times u(N)\xrightarrow{ad} u(N)$$
$$(v, i\rho) \mapsto ad_v(i\rho) = [v, i\rho]$$

where $[v, i\rho] = v(i\rho) - (i\rho)v$

The Exponential Map

$$\exp : u(N) \to U(N)$$
$$v \mapsto e^v = \sum_{k=0}^{\infty} \frac{v^k}{k!}$$

Moreover, $Ad_{\exp(v)} = \exp(ad_v)$, i.e.,

$$\begin{array}{ccc} v & \mapsto & ad_v \\ u(N) & \xrightarrow{ad} & End(u(N)) \\ \exp\downarrow & & \downarrow\exp \\ U(N) & \xrightarrow{Ad} & Aut(u(N)) \\ U & \mapsto & Ad_U \end{array}$$

The Special Orthogonal Group SO(3)

- The Lie group of rotations in $\mathbb{R}^3$ is

$$SO(3) = \{A \in GL(3,\mathbb{R}) : A^T = A^{-1} \text{ and } \det(A) = 1\}$$

- Its Lie algebra is

$$so(3) = \{A \in Mat(3,\mathbb{R}) : v^T = -v \text{ and } trace(v) = 0\}$$

- And we have

Dirac Belt Trick

$$\begin{array}{ccc} u(2) & \xrightarrow{ad} & End(u(2)) = so(3) \\ \exp\downarrow & & \downarrow\exp \\ U(2) & \xrightarrow{Ad} & Aut(u(3)) = SO(3) \end{array}$$

Chapter 4

Invariants of Quantum Entanglement

The Lie algebra $\ell(n)$ of the Local Group $L(n)$

Local Group

$$L(n) = \bigotimes_1^n SU(2) \subset U(2^n)$$

Big Adjoint action

$$L(n)\times u(2^n)\xrightarrow{Ad} u(2^n)$$

The Lie algebra $\ell(n)$ of $L(n)$ is the sub-Lie algebra of $u(2^n)$ generated by:

$$\left\{ \xi_{k_1 k_2 \cdots k_n} : \begin{array}{l} k_j \in \{0,1,2,3\}\ \forall j, \text{ and} \\ \text{where exactly one } k_j \neq 0 \end{array} \right\}$$

The Lie algebra $\ell(n)$ of the Local Group $L(n)$

For example,

- $\ell(1)$ is generated by $\{\xi_1, \xi_2, \xi_3\}$
- $\ell(2)$ by $\{\xi_{01}, \xi_{02}, \xi_{03}, \ \xi_{10}, \xi_{20}, \xi_{30}\}$
- $\ell(3)$ by

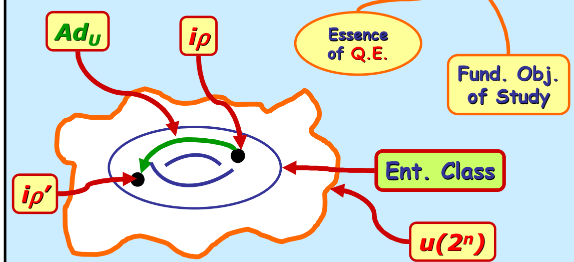
$$\{\xi_{001}, \xi_{002}, \xi_{003}, \ \xi_{010}, \xi_{020}, \xi_{030}, \ \xi_{100}, \xi_{200}, \xi_{300}\}$$

What it is all about

The essence of the RFPQE is to **understand** the **BIG Adjoint action** of the local group **$L(n)$** on the Lie algebra **$u(2^n)$**, namely, the action

What is a Q.E. invariant ?

We seek **invariants** of the **BIG Adjoint action** of the group **$L(n)$** on the Lie algebra **$u(2^n)$**. $L(n)\times u(2^n) \xrightarrow{Ad} u(2^n)$



The Algebra $C^\infty\left(u(2^n)\right)^{L(n)}$ of Q.E. Invariants

Let $C^\infty\left(u(2^n)\right) = \left\{ f : u(2^n) \to \mathbb{R} : f \text{ smooth} \right\}$

Definition. The **algebra of Q.E. invariants**, denoted by $C^\infty\left(u(2^n)\right)^{L(n)}$ is defined as

$$\left\{ f \in C^\infty\left(u(2^n)\right) : f\left(Ad_U(i\rho)\right) = f(i\rho)\ \forall U \in L(N) \right\}$$

A Complete Set of Invariants

If $f \in C^\infty\left(u(2^n)\right)^{L(n)}$, then

$$\therefore\quad i\rho \underset{loc}{\sim} i\rho' \Rightarrow f(i\rho) = f(i\rho')$$

$$\therefore\ f(i\rho) \neq f(i\rho') \Rightarrow i\rho \underset{not\ loc}{\sim} i\rho'$$

However,

$$f(i\rho) = f(i\rho') \Rightarrow i\rho \overset{?}{\underset{loc}{\sim}} i\rho'$$

In this case, we know nothing. The invariant is not enough to distinguish all Q.E. classes.

We seek **enough Q.E. invariants** to distinguish all Q.E. classes. Such a set of Q.E. invariants is called

A Complete Set of Invariants

How do we find Q.E. invariants ?

We seek $f : u(2^n) \to \mathbb{R}$ which are invariant under the BIG Adjoint action

$$L(n)\times u(2^n) \xrightarrow{Ad} u(2^n)$$

In other words, we seek f such that

$$f\left(Ad_U(i\rho)\right) = f(i\rho),\ \forall i\rho \in u(2^n),\ \forall U \in L(n)$$

Our Approach: Lift the problem to the Lie algebra $\ell(n)$ where it becomes a linear problem.

The BIG Adjoint action of $L(n)$ on $u(2^n)$ induces an **infinitesimal action**

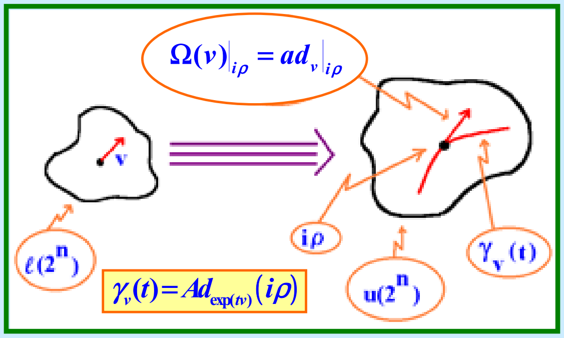
$$\ell(n) \xrightarrow{\Omega} Vec\left(u(2^n)\right)$$

Let $v \in \ell(n)$. We define the vector field $\Omega(v)$ by constructing the tangent vector $\Omega(v)\big|_{i\rho}$ for each $i\rho \in u(2^n)$. Let $\gamma_v(t)$ be a smooth curve in $u(2^n)$ defined by

$$\gamma_v(t) = Ad_{\exp(tv)}(i\rho)$$

Then $\gamma_v(t)$ passes through $i\rho$ at time, $t = 0$. Define $\Omega(v)\big|_{i\rho}$ to be the tangent vector to $\gamma_v(t)$ at time $t = 0$.

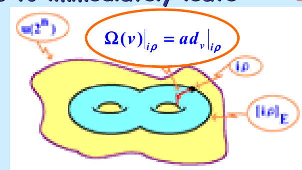
The Infinitesimal Action



What is the meaning of the infinitesimal action ?

$$\ell(n) \xrightarrow{\Omega} Vec\left(u(2^n)\right)$$

Each $\Omega(v)|_{i\rho}$ is a direction in $u(2^n)$ from which we can move w/o leaving the Q.E. class $[i\rho]_E$. Movement in all directions not in $\mathrm{Im}\left(\Omega(v)|_{i\rho}\right)$ will force us to immediately leave $[i\rho]_E$.



Objective 1. Find $Dim\,[i\rho]_E$ (Achieved !!!)

Consider $\ell(n) \xrightarrow[1-1]{\Omega} \mathrm{Im}(\Omega) \subset Vec\left(u(2^n)\right)$

Then $T_{i\rho}[i\rho]_E = \mathrm{Im}(\Omega)|_{i\rho} \subset Vec\left(u(2^n)\right)|_{i\rho} = T_{i\rho}u(2^n)$

Tangent Space to $[i\rho]_E$ at $i\rho$

Tangent Space to $u(2^n)$ at $i\rho$

Hence,

$$Dim\,[i\rho]_E = Dim\,T_{i\rho}[i\rho]_E = Dim\left(\mathrm{Im}\,\Omega|_{i\rho}\right)$$

Obj. 2. Find a complete set of Q.E. invariants

Achieved !!

$$\ell(n) \xrightarrow[1-1]{\Omega} \mathrm{Im}(\Omega) \subset Vec\left(u(2^n)\right) = Der\left(C^\infty u(2^n)\right)$$

Recall that $\mathrm{Im}(\Omega)$ consists of all directions in $u(2^n)$ that we can move in w/o leaving a Q.E. class that we are presently in. If

$$f \in \left(C^\infty u(2^n)\right)^{L(n)}$$

then f will not change if we move in any direction in $\mathrm{Im}(\Omega)$.

The above approach leads to the following theorem:

Theorem. Let $v_1, v_2, \ldots, v_{3n}$ be a vector space basis of the Lie algebra $\ell(n)$. Then

$$f \in \left(C^\infty u(2^n)\right)^{L(n)} \Leftrightarrow \Omega(v_j)\,f = 0\ \forall j$$

Hence, finding Q.E. invariants is equivalent to solving the above system of linear partial differential equations.

Chapter 5

Q.E. Invariants for *n=1* Qubits

Q.E. Invariants for *n=1* Qubits

- This is a trivial, but instructive case
- No Q.E.; but $\exists$ many Q.E. classes !!!

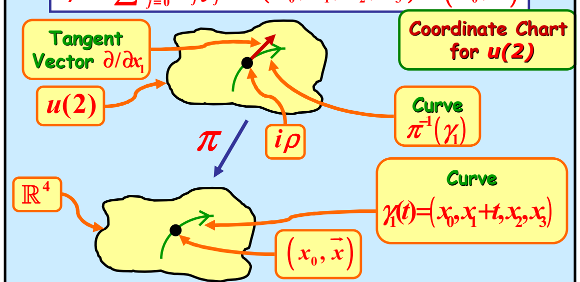
- $L(1) = SU(2) = $ 3-sphere
- $\ell(1) = su(2) = \mathbb{R}^3$
- $i\rho \in u(2) = \mathbb{R}^4$

For $\ell(1)$, basis $= \left\{ \xi_1 = -\frac{i\sigma_1}{2}, \xi_2 = -\frac{i\sigma_2}{2}, \xi_3 = -\frac{i\sigma_3}{2} \right\}$

For $u(2)$, basis $= \{\xi_0, \xi_1, \xi_2, \xi_3\}$

$$u(2) \xrightarrow{\pi} \mathbb{R}^4$$

$$i\rho = \sum_{j=0}^{3} x_j \xi_j \mapsto (x_0, x_1, x_2, x_3) = (x_0, \vec{x})$$



$\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}$ basis for $Vec(u(2))$

We identify $u(2) = \mathbb{R}^4$ via the chart π

$$\therefore i\rho = x_0\xi_0 + x_1\xi_1 + x_2\xi_2 + x_3\xi_3 = (x_0, \vec{x})$$

$$\therefore ad_{\xi_k} = 0 \oplus L_k = \begin{pmatrix} 0 & 0 \\ 0 & L_k \end{pmatrix}$$

where

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are the infinitesimal generators of the Lie algebra $so(3)$ of the special orthogonal group $SO(3)$

Hence,

$$ad_{\xi_k}(i\rho) = \begin{pmatrix} 0 & 0 \\ 0 & L_k \end{pmatrix} \begin{pmatrix} x_0 \\ \vec{x} \end{pmatrix} = L_k \vec{x}$$

and so

$$\Omega(\xi_k) = ad_{\xi_k}(i\rho) \bullet \begin{pmatrix} \partial/\partial x_0 \\ \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{pmatrix} = \left(L_k \vec{x}^{\,T}\right) \bullet \nabla$$

where

$$\nabla = (\partial/\partial x_1, \partial/\partial x_1, \partial/\partial x_1)^T$$

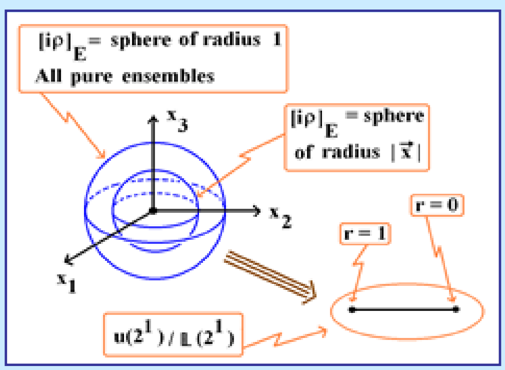
Thus, $\mathrm{Im}(\Omega)$ is spanned by

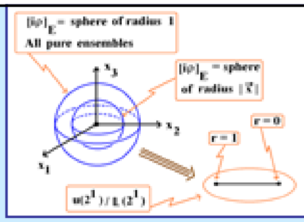
$$\begin{cases} \Omega(\xi_1) = x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} \\ \Omega(\xi_2) = x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1} \\ \Omega(\xi_3) = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} \end{cases}$$

Hence,

$$Dim\left[(\mathrm{Im}\,\Omega)\big|_{i\rho} \right] = \begin{cases} 2 & \text{if } \vec{x} \neq 0 \\ 0 & \text{if } \vec{x} = 0 \end{cases}$$

The Bloch "Sphere"





So $\Omega(\xi_1)|_{i\rho}, \Omega(\xi_2)|_{i\rho}, \Omega(\xi_3)|_{i\rho}$ span the tangent space to $[i\rho]_E$ at $i\rho$

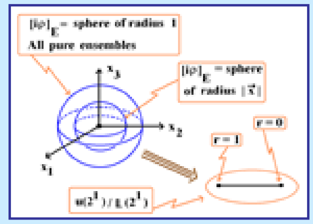
Moreover, $x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}$

is the normal vector field to $[i\rho]_E$

Finally, a Complete Set of Invariants

The solution of

$$\begin{cases} \Omega(\xi_1) f = 0 \\ \Omega(\xi_2) f = 0 \\ \Omega(\xi_3) f = 0 \end{cases}$$



is

$$f(i\rho) = \left|\vec{x}\right|^2$$

Chapter 6

Q.E. Invariants for *n*=2 Qubits

Q.E. Invariants for *n*=2 Qubits

- $L(2) = SU(2) \otimes SU(2)$
- $\ell(2) = su(2) \boxplus su(2) = \mathbb{R}^6$
- $i\rho \in u(2H2) = \mathbb{R}^{16}$

where

Kronecker Sum

$$A \boxplus B \equiv (A \otimes I) \oplus (I \otimes B)$$

For $\ell(2)$, basis $= \{\xi_{01}, \xi_{02}, \xi_{03} \quad \xi_{10}, \xi_{20}, \xi_{30}\}$

For $u(2^2)$, basis $= \{\xi_{jk} : j,k \in \{0,1,2,3\}\}$

Coordinate Chart for *u(2²)*

$$u(2^2) \xrightarrow{\pi} \mathbb{R}^{16}$$

$$i\rho = \sum_{j,k=0}^{3} x_{jk}\xi_{jk} \mapsto (x_{00}, x_{01}, x_{03}, x_{10}, \ldots, x_{33})$$

$$= \left(x_{00}, \vec{x}_{0*}, \vec{x}_{*0}, \vec{x}_{**}\right) = \begin{pmatrix} x_{00} & \vec{x}_{0*} \\ \vec{x}_{*0}^{\,T} & x_{**} \end{pmatrix}$$

where

$\vec{x}_{0*} = (x_{01}, x_{02}, x_{03})$	$x_{**} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$
$\vec{x}_{0*} = (x_{10}, x_{20}, x_{30})$	
$\vec{x}_{**} = (x_{11}, x_{12}, x_{13}, \quad x_{21}, x_{22}, x_{23}, \quad x_{31}, x_{32}, x_{33})$	

A basis for $Vec\left(u(2^2)\right)$ is

$$\left\{ \frac{\partial}{\partial x_{jk}} : j,k = 0.1,2,3 \right\}$$

$$\begin{cases} ad_{\xi_{0k}} = \left(0 \oplus ad_{\xi_k}\right) \otimes I_4 = \left(0 \oplus L_k\right) \otimes I_4 \\ ad_{\xi_{k0}} = I_4 \otimes \left(0 \oplus ad_{\xi_k}\right) = I_4 \otimes \left(0 \oplus L_k\right) \end{cases}$$

So,

$$ad_{\xi_{0k}}(i\rho) = \left(\left(0 \oplus L_k\right) \otimes I_4 \right) \begin{pmatrix} x_{00} \\ \vec{x}_{0*}^{\,T} \\ \vec{x}_{*0}^{\,T} \\ \vec{x}_{**}^{\,T} \end{pmatrix}$$

A similar formula holds for $ad_{\xi_{k0}}(i\rho)$

Hence,

$$\Omega(\xi_{0k}) = ad_{\xi_{0k}}(i\rho) \bullet \nabla$$
$$\Omega(\xi_{k0}) = ad_{\xi_{k0}}(i\rho) \bullet \nabla$$

where

$$\nabla = \left(\frac{\partial}{\partial \vec{x}_{0*}}, \frac{\partial}{\partial \vec{x}_{*0}}, \frac{\partial}{\partial \vec{x}_{0*}}, \frac{\partial}{\partial \vec{x}_{**}} \right)$$

$\mathrm{Im}(\Omega)$ is spanned by

$$\begin{cases} \Omega(\xi_{01}) = x_{02}\frac{\partial}{\partial x_{03}} - x_{03}\frac{\partial}{\partial x_{02}} + \sum_j \left(x_{2j}\frac{\partial}{\partial x_{3j}} - x_{3j}\frac{\partial}{\partial x_{2j}} \right) \\ \Omega(\xi_{02}) = x_{03}\frac{\partial}{\partial x_{01}} - x_{01}\frac{\partial}{\partial x_{03}} + \sum_j \left(x_{3j}\frac{\partial}{\partial x_{1j}} - x_{1j}\frac{\partial}{\partial x_{3j}} \right) \\ \Omega(\xi_{03}) = x_{01}\frac{\partial}{\partial x_{02}} - x_{02}\frac{\partial}{\partial x_{01}} + \sum_j \left(x_{1j}\frac{\partial}{\partial x_{2j}} - x_{2j}\frac{\partial}{\partial x_{1j}} \right) \end{cases}$$

$$\begin{cases} \Omega(\xi_{10}) = x_{20}\frac{\partial}{\partial x_{30}} - x_{30}\frac{\partial}{\partial x_{20}} + \sum_j \left(x_{j2}\frac{\partial}{\partial x_{j3}} - x_{j3}\frac{\partial}{\partial x_{j2}} \right) \\ \Omega(\xi_{20}) = x_{30}\frac{\partial}{\partial x_{10}} - x_{10}\frac{\partial}{\partial x_{30}} + \sum_j \left(x_{j3}\frac{\partial}{\partial x_{j1}} - x_{j1}\frac{\partial}{\partial x_{j3}} \right) \\ \Omega(\xi_{30}) = x_{10}\frac{\partial}{\partial x_{20}} - x_{20}\frac{\partial}{\partial x_{10}} + \sum_j \left(x_{j1}\frac{\partial}{\partial x_{j2}} - x_{j2}\frac{\partial}{\partial x_{j1}} \right) \end{cases}$$

It follows that, $Dim\left[\left(\mathrm{Im}\,\Omega\right)\big|_{i\rho} \right] = 6$ a.e.

In other words, almost all density operators $i\rho$ belong to a 6 dimensional Q.E. class.

But ... there are some important **exceptions**.

Please note that all four 2-qubit Bell states lie in the same Q.E. class. Because of this, teleportation is possible.

Consider the Bell state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

So,

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

In terms of the basis $\{\xi_{jk}\}$, this becomes

$$i\rho = \frac{1}{4}\left[(-1)\xi_{00} + (-1)\xi_{11} + (+1)\xi_{22} + (+1)\xi_{33}\right]$$

Thus, in this case $\mathrm{Im}(\Omega)\big|_{i\rho}$ is spanned by

$$\begin{cases} \Omega(\xi_{01})\big|_{i\rho} = \partial/\partial x_{32} + \partial/\partial x_{23} \\ \Omega(\xi_{02})\big|_{i\rho} = -\partial/\partial x_{13} + \partial/\partial x_{31} \\ \Omega(\xi_{03})\big|_{i\rho} = -\partial/\partial x_{21} - \partial/\partial x_{12} \end{cases}$$

$$\begin{cases} \Omega(\xi_{10})\big|_{i\rho} = \partial/\partial x_{23} + \partial/\partial x_{32} \\ \Omega(\xi_{20})\big|_{i\rho} = -\partial/\partial x_{31} + \partial/\partial x_{13} \\ \Omega(\xi_{30})\big|_{i\rho} = -\partial/\partial x_{12} - \partial/\partial x_{21} \end{cases}$$

Hence, the dimension of the Bell state entanglement class $[i\rho_{Bell}]_E$ is an exceptional ***3***, i.e.,

$$Dim[i\rho_{Bell}]_E = Dim\left[(\mathrm{Im}\,\Omega)\big|_{i\rho_{Bell}}\right] = 3$$

A Complete Set of Q.E. Invariants for *n=2* Qubits

Linden, Popescu, Sudberry

Let us use the above mentioned chart π to make the identification

$$i\rho = \begin{pmatrix} x_{00} & \vec{x}_{0*} \\ \vec{x}_{*0}^{\,T} & x_{**} \end{pmatrix}$$

and let

$$Z = x_{**}x_{**}^T$$

A Complete Set of Q.E. Invariants for *n=2* Qubits

The following set of ***9*** algebraically independent polynomial functions form a <u>basic set of polynomial Q.E. invariants</u>:

$trace(Z)$	$trace(Z^2)$	$\det(x_{**})$
$x_{0*}x_{0*}^T$	$x_{0*}Zx_{0*}^T$	$x_{0*}Z^2x_{0*}^T$
$x_{0*}x_{**}x_{*0}^T$	$x_{0*}Zx_{**}x_{*0}^T$	$x_{0*}Z^2x_{**}x_{*0}^T$

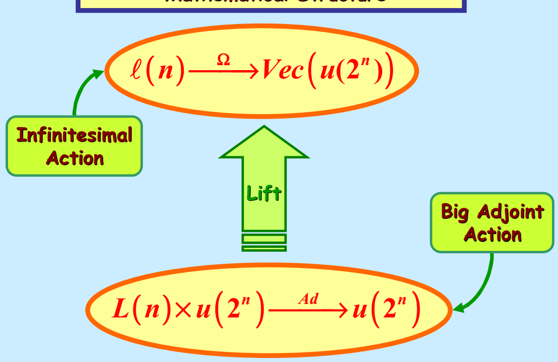
But … they do not form a complete set of Q.E. invariants !!!

A tenth polynomial $x_{0*}\bullet(Zx_{0*}^T)\times(Zx_{0*}^T)$, which is algebraically dependent on the above polynomials is need to determine the sign of the components of $i\rho$ and to form a <u>complete set of Q.E. invariants</u>.

Chapter 7

Conclusion

The RFPQE "Lives" in the Following Mathematical Structure



The General Problem of Finding a Complete Set of Q.E. Invariants for *n* Qubits

In the paper

"An Entangled Tale of Quantum Entanglement,"

it is shown how the general problem of finding a complete set of Q.E. invariants for ***n*** qubits can be reduced to the task of finding a complete set of solutions to the system of ***3n*** PDEs listed on the following slide:

For a Complete Set of Q.E. Invariants Solve the Following System of 3n PDEs

General *n* Qubit Case

$$\begin{cases} \sum_{j_1, j_2, \ldots, j_{n-1}=0}^{3} x_{* j_1 j_2 \cdots j_{n-1}} \times \frac{\partial f}{\partial x_{* j_1 j_2 \cdots j_{n-1}}} = 0 \\ \sum_{j_1, j_2, \ldots, j_{n-1}=0}^{3} x_{j_1 * j_2 \cdots j_{n-1}} \times \frac{\partial f}{\partial x_{j_1 * j_2 \cdots j_{n-1}}} = 0 \\ \vdots \\ \sum_{j_1, j_2, \ldots, j_{n-1}=0}^{3} x_{j_1 j_2 \cdots j_{n-1} *} \times \frac{\partial f}{\partial x_{j_1 j_2 \cdots j_{n-1} *}} = 0 \end{cases}$$

Where, for example,

$$x_{* j_1 j_2 \cdots j_{n-1}} \times \frac{\partial f}{\partial x_{* j_1 j_2 \cdots j_{n-1}}}$$

denotes the vector cross product of the two vectors

$x_{* j_1 j_2 \cdots j_{n-1}}$ and $\frac{\partial f}{\partial x_{* j_1 j_2 \cdots j_{n-1}}}$

Problem: The Number of Q.E. Invariants Grows Exponentially with the Number n of Qubits !

Qubits n	$Max\ Dim[i\rho]_E$ $3n$	$\#\ CSQEI$ 4^n-1-3n
2	6	9
3	9	54
4	12	243
5	15	1008
⋮	⋮	⋮

The RFPQE Is Difficult for $n \geq 3$ Qubits

Some progress has been made for 3 and 4 qubits. For example, Meyer & Wallach have recently been able to count the number of entanglement classes for n=4 qubits.

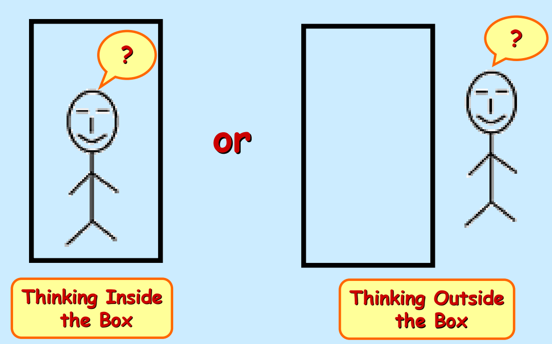
There is More to Q.E. Than the RFPQE

The Big Adjoint action does not mathematically fully captures all of the physical phenomenon of Q.E.

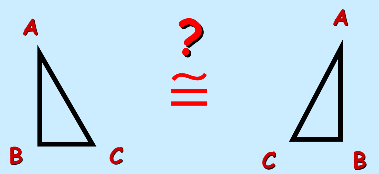
For example, the mathematical model of the Big Adjoint action needs to be extended to capture such other physical effects of Q.E. as:

- The effects of classical communication (LOCC)
- The distillation of entangled states
- And much more …

Thinking Inside or Outside the Box ?

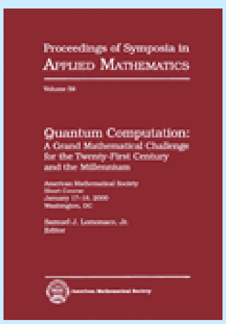


Thinking Inside or Outside the Box ?



3-D Physical Euclidean Geometry

Quantum Computation: A Grand Mathematical Challenge for the Twenty-First Century and the Millennium, **Samuel J. Lomonaco, Jr. (editor)**, AMS PSAPM/58, (2002).



Quantum Computation and Information, **Samuel J. Lomonaco, Jr. and Howard E. Brandt (editors)**, AMS CONM/305, (2002).