Preface

Years ago, when I was a beginning and naive graduate student, eagerly studying mathematics at Princeton, I wanted to know the secret to success in mathematics. I actually did believe that there must be just one secret that would lead to fame and fortune in this field. So during tea time in Old Fine Hall, I approached my all-wise thesis advisor (then reverently called Professor Fox), and naively posed the question:

> *"What is the secret to success in mathematics?"*

Ralph Fox, without a moment's hesitation, fired back:

> *"Work where two research fields are merging."*

At that time, I did not fully comprehend the meaning and the significance of his prompt, laconic answer. But as the years have passed, I have begun to understand more fully the wisdom behind his terse one-line response. His words of wisdom have indelibly left an impression that has shaped the many choices made throughout my research career.

Perhaps, Ralph Fox was referring to his student John Milnor's success in creating the field of differential topology, a beautiful merger of differential geometry and topology. Or perhaps, he was referring to Stephen Smale's proof of the higher dimensional Poincare conjecture? Or was it his student John Stallings' creative merger of group theory and topology? In any case, throughout my career I have come to see Ralph Fox's prophecy repeatedly come true, over and over again.

Now, Ralph Fox's prophecy is becoming true once again in the newly emerging field of quantum computation and quantum information, i.e., quantum information science (QIS). Never before have I found such a convergence of so many research fields that are currently shaping the development of QIS, … and yes, of mathematics, itself. Never before has there been such a rich and immense research opportunity for the mathematical community. In particular, mathematics is now shaping QIS, and in turn, QIS is now shaping the development of mathematics.

For that reason, I organized and gave an AMS Short Course on Quantum Computation at the Annual meeting of the American Mathematical Society held in Washington, DC in January of 2000. This past short course is now recorded and encapsulated in the AMS book

> **"Quantum Computation: A Grand Mathematical Challenge for the Twenty-First Century and the Millenium,"** PSAPM, vol. 58, Providence, RI, (2002).

For the same reason, I also organized at the same AMS meeting an AMS Special Session on Quantum Computation and Information which has been recorded in a second AMS book

"Quantum Computation and Information," AMS, CONM, vol. 305, (2002).

Nine years later, much has changed. But one thing still remains unchanged. The mathematical opportunities are still there, more than ever before. For that reason, I once again organized and gave an AMS Short Course on QIS at the Annual Meeting of the American Mathematical Society held in Washington, DC in January of 2009. This AMS Short Course is now recorded in this AMS PSAPM volume.

Unlike the previous volume, which emphasized quantum algorithms, this new volume instead emphasizes quantum information and its contributions to many new resulting developments in mathematical research. For the reader with little, if any, background in or knowledge of quantum mechanics, we have provided in the appendix of this book a reprint of the paper by Samuel Lomonaco, entitled **"A Rosetta Stone for Quantum Mechanics with an Introduction to Quantum Computation,"** originally published in the 2000 AMS Short Course volume.

Much like the 2009 AMS Short Course, this volume is naturally divided into two parts.

In Part 1, two papers provide an overview of some of the latest developments in the theory of quantum information. The first paper by Patrick Hayden entitled, **"Concentration of Measure Effects in Quantum Computation,"** gives a survey of quantum information theory, i.e., a survey of the generalization and application of classical information theory to quantum communication and computation. The result is a rich theory of surprising simplicity. The quantum mechanical formalisms of density operators, partial traces, and super-operators are introduced, and then used to discuss the quantum communication channels. The paper ends with a discussion of how entanglement concentration of measure naturally occurs in higher dimensional subspaces.

The second paper by Daniel Gottesman, entitled **"Quantum Error Correction and Fault Tolerance,"** gives an introduction to QIS's first line of defense against the ravages of quantum decoherence, i.e., quantum error correcting codes and fault tolerance. The paper begins with a discussion of error models, moves on to quantum error correction and the stabilizer formalism, then to fault tolerant circuits and thresholds, and ends with a quantum error correction sonnet.

Part 2 consists of four papers illustrating how quantum information can be used as a vehicle for creating new mathematics as well as new developments in QIS. The first paper by Howard E. Brandt entitled, **"Riemannian Geometry of Quantum Computation,"** is an introduction to recent developments in the application of differential geometry to quantum algorithm implementation. In particular, it shows how Riemannian Geometry can be used to efficiently factor (i.e., compile) a global unitary transformation (i.e., quantum computer program fragment) into a product of local unitary transformations (i.e., into a sequence of quantum computer assembly language instructions.)

The second paper by Louis H. Kauffman and Samuel J. Lomonaco, Jr. entitled, **"Topology and Quantum Computing,"** gives a survey of a number of connections between quantum topology and quantum computation. The paper begins with a question about the possibility of a connection between topological and quantum entanglement. It then discusses the unitary solutions of the Yang-Baxter equation as universal quantum gates and as representations of the braid group. The bracket polynomial is presented in the context of Temperley-Lieb recoupling and topological computation. An analysis of the Fibonacci model is given. Finally, quantum algorithms for the colored Jones polynomial and the Witten-Reshetikhin-Turaev invariant are discussed.

The remaining two papers by Samuel J. Lomonaco and Louis H. Kauffman entitled respectively, **"Quantum Knots and Mosaics"** and **"Quantum Knots and Lattices: A Blueprint for Quantum Systems that Do Rope Tricks"** together illustrate how mathematics and quantum physics can be seamlessly interwoven into one fabric to create new research directions in both of these two fields. The first paper is based on the Reidemeister knot moves, the second on a new set of knot moves called wiggle, wag, and tug. The objective of each of these papers is to construct a blueprint for a physically implementable quantum system that simulates (and hence contains) tame knot theory. Each accomplishes this in three steps. In step one, tame knot theory is found to be equivalent to a formal writing system, i.e., a formal computer programming language. In step two, the formal rewriting system is then found to be equivalent to a group representation. Finally, in step three, the group representation is used to define a quantum knot system $(\mathcal{K},\Lambda)$, consisting of a Hilbert space $\mathcal{K}$ of quantum knots together with a unitary group Λ of quantum knot moves. Knot invariants now become quantum observables, physically measurable in a laboratory setting. The dynamic behavior of knots can now be expressed in terms of Hamiltonians. Most surprisingly, the knot moves can be transformed into infinitesimal moves which can be used to create knot variational derivatives. An immediate consequence is that now knot invariants can be defined as those knot functionals with vanishing knot variational derivatives.

It is hoped that this book will encourage mathematicians to take advantage of the many research opportunities in quantum information science.

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