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An Unentangled Gleason's Theorem

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ABSTRACT. The purpose of this note is to give a generalization of Gleason's theorem inspired by recent work in quantum information theory. For multipartite quantum systems, each of dimension three or greater, the only nonnegative frame functions over the set of unentangled states are those given by the standard Born probability rule. However, if one system is of dimension 2 this is not necessarily the case.

1. Introduction.

Let H be a Hilbert space with unit sphere $S(H)$. Following Gleason ([Gleason]) we will call a function $f : S(H) \to \mathbb{C}$ a frame function of weight w if for every orthonormal basis $\{v_i\}$ of $S(H)$

$$\sum_i f(v_i) = w. \tag{1}$$

In [Gleason] the following theorem was proved

THEOREM 1. *If* $\dim H \geq 3$ *and* f *is a frame function that takes non-negative real values then there exists a self adjoint trace class operator* $T : H \to H$ *such that*

$$f(v) = \langle v|T|v\rangle, v \in S(H).$$

This theorem is of importance to quantum mechanics because it allows a significant weakening of the axioms, showing that the Born probability rule [Born] provides the unique class of probability assignments for measurement outcomes so long as those probabilities are specified by frame functions [Pitowsky]. The theorem also rules out a large class of hidden-variable explanations for quantum statistics, the so-called noncontextual hidden variables, in dimension 3 or greater. The interested reader should consult [Bell] for a discussion of this point. If the Hilbert space is of dimension 2, then the statement in the theorem is easily seen to be false.

The purpose of this note is to give a generalization of Gleason's theorem inspired by recent work in quantum information theory. In that context the issue of *local* measurements and operations on multipartite quantum systems (as opposed to the full set of operations) is of the utmost importance [BDFMRSSW]. For instance,

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it has been pointed out that probabilities for the outcomes of local measurements are enough to uniquely specify the quantum state from which they arise if the field of the Hilbert space is complex, though this fails for real and quaternionic Hilbert spaces [Araki,Wootters].

Chris Fuchs has asked to what extent local and semi-local measurements not only uniquely specify the quantum state, but also a Born-like rule as in Gleason's result [Fuchs]. In this regard, the following formalization appears natural. We confine our attention to finite dimensional Hilbert spaces for the sake of simplicity. Let $H_1, ..., H_n$ be Hilbert spaces. Set $H = H_1 \otimes H_2 \otimes \cdots \otimes H_n$. Let $\Sigma = \Sigma(H_1, ..., H_n)$ denote the subset of $S(H)$ consisting of those elements of the form $a_1 \otimes \cdots \otimes a_n$ with $a_i \in S(H_i)$ for $i = 1, ..., n$. In the jargon of quantum information theory such states are called *unentangled* or *product states*. The ones that are not of this form are said to be *entangled*. An orthonormal basis $\{v_i\}$ of H is said to be unentangled if $v_i \in \Sigma$ for all i. We say that $f : \Sigma \to \mathbb{C}$ is an unentangled frame function of weight w if whenever $\{v_i\}$ is an unentangled orthonormal basis of H then f satisfies (1) above. We establish the following result.

THEOREM 2. *If* $\dim H_i \geq 3$ *for all* i *and if* $f : \Sigma \to \mathbb{R}$ *is a non-negative unentangled frame function then there exists* $T : H \to H$ *a self adjoint trace class operator such that* $f(v) = \langle v|T|v\rangle$ *for all* $v \in \Sigma$.

This theorem is an almost direct consequence of Gleason's original theorem. We will give a proof of it in the next section. The second result in this paper shows that the dimensional condition is necessary.

It should be noted however, that despite the absence of entangled or "nonlocal" states in Σ, in [BDFMRSSW] it is asserted that not all unentangled bases correspond to quantum measurements that can be carried out by local means alone (even with iterative procedures based on weak local measurements and unlimited amounts of classical communication between the measurers at each site). The simplest kind of purely local measurement is given by an alternative type of basis adapted to the tensor product structure. This is a *product basis* and is defined as to be a basis of the form $\{u_{i_1 1} \otimes u_{i_2 2} \otimes \cdots \otimes u_{i_n n}\}$ where $u_{1j}, ..., u_{n_j j}$ is an orthonormal basis of H_j. We could define a product frame function in the same way as we did for an unentangled frame function except that we only assume that there exists a weight w such that $\sum_{i_1,i_2,...,i_n} f(u_{i_1 1} \otimes u_{i_2 2} \otimes \cdots \otimes u_{i_n n}) = w$ for every product basis. One can ask whether this is all that us necessary for the conclusion of the theorem above. The answer is no and a method of "finding" a large class of examples will be given at the end of the next section (see the proposition at the end of the section). This result amasses some evidence that the structure of local measurements alone is not enough to establish the Born rule for multipartite systems, but a full answer would require consideration of the largest class of local measurements in [BDFMRSSW].

These issues also spawn another theorem.

THEOREM 3. *Let* $\dim H_1 = 2$ *and let* $f : S(H_1) \to \mathbb{C}$ *be a frame function of weight* w_1 *and* $g : \Sigma(H_2, ..., H_n) \to \mathbb{C}$ *be an unentangled frame function of weight* w_2*. We set* $h(v_1 \otimes u) = f(v_1)g(u)$ *for* $u \in \Sigma(H_2, ..., H_n)$*. Then* h *is an unentangled frame function of weight* $w_1 w_2$*.*

This result is a bit harder and the proof involves a method (see Theorem 5) that describes a combinatorial scheme for finding all unentangled orthonormal bases where all of the spaces, H_i, have dimension 2. This analysis in turn leads to a natural question. Given and unentangled orthonormal set can it be extended to an unentangled orthonormal basis? Or even stronger: Can it be a proper subset of an unentangled orthonormal set? This question was studied in [BDMSST]. We conclude the paper by giving a proof based on simple algebraic geometry of the following theorem which is related to the bound that occurs in [BDMSST].

THEOREM 4. *Let V be a subspace of $H_1 \otimes \cdots \otimes H_n$ such that if $v \in V$ and $v \neq 0$ then v is entangled. Then* $\dim V \leq \dim(H_1) \cdots \dim(H_n) - \sum(\dim H_i - 1) - 1$. *Furthermore, the upper bound is attained.*

2. The unentangled Gleason theorem.

In this section we will give a proof of Theorem 2. If $n = 1$ the statement is just Gleason's theorem. We consider the situation of $H = H_0 \otimes V$ with $V = H_1 \otimes H_2 \otimes \cdots \otimes H_n$ and $\dim H_i \geq 3$ for all i. We prove Theorem 1 by induction (i.e. assume the result for n). We note that if $\{v_i\}$ is an orthonormal basis of H_0 and if for each i, $\{u_{ij}\}$ is an unentangled orthonormal basis of V then the set $\{v_i \otimes u_{ij}\}$ is an unentangled orthonormal basis of H. Thus if w is the weight of f then we have

$$\sum_j f(v_1 \otimes u_{1j}) = w - \sum_{i \geq 2, j} f(v_i \otimes u_{ij}).$$

Thus for each $v \in S(H_0)$ the function $f_v(u) = f(v \otimes u)$ is an unentangled frame function. The inductive hypothesis implies that for each $v \in S(H_0)$ there exists a self adjoint (due to the reality of f) linear operator $T(v)$ such that $f(v \otimes u) = \langle u|T(v)|u\rangle$ for $u \in \Sigma(H_1, ..., H_n)$. Similarly, if $\{u_i\}$ is an unentangled orthonormal basis of V and for each i, $\{v_{ij}\}$ is an orthonormal basis of H_0 then $\{v_{ij} \otimes u_i\}$ is an unentangled orthonormal basis of H. We therefore conclude as above that if $u \in \Sigma(H_1, ..., H_n)$ then there exists $S(u)$ a self adjoint linear operator on H_0 so that $f(v \otimes u) = \langle v|S(u)|v\rangle$ for all $v \in H_0$.

Let $\{u_i\}$ be an unentangled orthonormal basis of V and let $\{v_j\}$ be an orthonormal basis of H_0. Set

$$a_{ij}(v) = \langle u_i|T(v)|u_j\rangle$$

and

$$b_{ij}(u) = \langle v_i|S(u)|v_j\rangle, u \in \Sigma(H_1, ..., H_n).$$

We now observe that if $v = \sum_i x_i v_i$ and if $u = \sum_j y_j u_j$ then we have

$$\sum_{p,q} a_{p,q}(v)\bar{y}_p y_q = \sum_{r,s} b_{r,s}(u)\bar{x}_r x_s.$$

If we substitute $v = v_r$ then we have

$$b_{rr}(u) = \sum_{p,q} a_{p,q}(v_r)\bar{y}_p y_q.$$

Now assuming that $r \neq s$ and taking $v = \frac{1}{\sqrt{2}}(v_r + v_s)$ we have

$$\operatorname{Re} b_{rs}(u) = \sum_{p,q} a_{p,q}(\frac{1}{\sqrt{2}}(v_r + v_s))\bar{y}_p y_q - \frac{1}{2}\left(\sum_{p,q} a_{p,q}(v_r)\bar{y}_p y_q + \sum_{p,q} a_{p,q}(v_s)\bar{y}_p y_q\right).$$

Also if we take $v = \frac{1}{\sqrt{2}}(v_r + iv_s)$ then we have

$$-\operatorname{Im} b_{rs}(u) = \sum_{p,q} a_{p,q}(\frac{1}{\sqrt{2}}(v_r + iv_s)) y_p \overline{y}_q - \frac{1}{2}\left(\sum_{p,q} a_{p,q}(v_r) y_p \overline{y}_q + \sum_{p,q} a_{p,q}(v_s) y_p \overline{y}_q\right).$$

Thus if we set

$$c_{rrpq} = a_{pq}(v_r)$$

and if $r \neq s$ then

$$c_{rspq} = a_{p,q}\left(\frac{1}{\sqrt{2}}(v_r + v_s) - \frac{1}{2}(a_{p,q}(v_r) + a_{p,q}(v_s))\right) + a_{p,q}\left(\frac{1}{\sqrt{2}}(v_r + iv_s) - \frac{1}{2}(a_{p,q}(v_r) + a_{p,q}(v_s))\right)$$

Then

$$f(v \otimes u) = \sum_{rspq} c_{rspq} \bar{x}_r \bar{y}_p x_s y_q.$$

This is the content of the theorem.

We will now give a counterexample to the analogous assertion for product bases.

Proposition 5. *Let H_1 and H_2 be finite dimensional Hilbert spaces of dimension greater than 1. Then there exists $f : \Sigma(H_1, H_2) \to [0, \infty)$ such that $\sum_{i,j} f(u_i \otimes v_j) = w$, with $w \in \mathbb{R}$ fixed, for all choices $\{u_i\}$ and $\{v_j\}$ of orthonormal bases of H_1 and H_2 respectively but there is no linear endomorphism, T, on $H_1 \otimes H_2$ such that $f(u \otimes v) = \langle u \otimes v | T | u \otimes v \rangle$ for $u \in S(H_1)$ and $v \in S(H_2)$.*

Proof. Let for $w > 0$, $\mathcal{P}_w$ denote the set of all Hermitian positive semi-definite endomorphisms, A, of H_2 such that $tr(A) = w$. Fix $w_o = \frac{w}{\dim H_1}$. Let $\varphi : S(H_1) \to \mathcal{P}_{w_o}$ be a mapping (completely arbitrary). Set $f(u \otimes v) = \langle v | \varphi(u) | v \rangle$, for $u \in S(H_1)$ and $v \in S(H_2)$. If $\{u_i\}$ is an orthonormal basis of H_1 and if $\{v_j\}$ is an orthonormal basis of H_2 then

$$\sum_{i,j} f(u_i \otimes v_j) = \sum_i \left(\sum_j \langle v_j | \varphi(u_i) | v_j \rangle\right) = \sum_i tr(\varphi(u_i)) = \dim(H_1) w_o.$$

Note: In this argument only one factor need be finite dimensional. Also note that f can be chosen to be continuous.

3. Unentangled Bases

In this section we will develop the material on "unentangled bases" that we will need to prove Theorem 3 (in fact as we shall see a generalization). Let V be a 2-dimensional Hilbert space and let H be an n-dimensional Hilbert space. Fix $\Sigma \subset S(H)$ such that $\lambda\Sigma = \Sigma$ for all $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. We will use the notation $S(V) \otimes \Sigma = \{v \otimes w | v \in S(V), w \in \Sigma\}$.

If $a \in S(V)$ then up to scalar multiple there is exactly one element of $S(V)$ that is perpendicular to a. We will denote a choice of such an element by $\widehat{a}$. The main result of this section is

THEOREM 6. *If $\{u_j\}_{j=1}^{2n}$ is an orthonormal basis of $V \otimes H$ with $u_j \in S(V) \otimes \Sigma$ for $j = 1, ..., 2n$ then there exists a partition*

$$n_1 \geq n_2 \geq ... \geq n_r > 0$$

of n, an orthogonal decomposition

$$H = U_1 \oplus ... \oplus U_r,$$

elements $a_1, ..., a_r \in S(H)$, and for each $i = 1, ..., r$ orthonormal bases $\{b_{i1}, ..., b_{in_i}\}$ and $\{c_{i1}, ..., c_{in_i}\}$ of U_i such that

$$\{u_i | \, i = 1, ..., 2n\} = \bigcup_{i=1}^{r} \left(\{a_i \otimes b_{ij} | \, j = 1, ..., n_i\} \cup \{\widehat{a}_i \otimes c_{ij} | \, j = 1, ..., n_i\} \right).$$

Before we prove the theorem we will make several preliminary observations. Let $\{u_i\}$ be as in the statement of the theorem. Then each $u_i = a_i \otimes h_i$ with $a_i \in S(V)$ and $h_i \in \Sigma$.

1. For each i there exists j such that a_j is a multiple of $\widehat{a}_i$.

If not then we would have $\langle a_i | a_j \rangle \neq 0$ for all j. Since $\langle a_i \otimes h_i | a_j \otimes h_j \rangle = \langle a_i | a_j \rangle \langle h_i | h_j \rangle$, $\langle h_i | h_j \rangle = 0$ for all $j \neq i$. This implies that $\{u_j\}_{j \neq i} \subset V \otimes \{h_i^{\perp}\}$. This space has dimension equal to $2(n-1)$. So it could not contain $2n-1$ orthonormal elements. This contradiction implies that assertion 1. is true.

2. Assume that $i \neq j$. If $\langle a_i | a_j \rangle \neq 0$ then $\langle h_i | h_j \rangle = 0$. If $\langle h_i | h_j \rangle \neq 0$ then $\langle a_i | a_j \rangle = 0$.

This is clear (see the proof of 1.)

We will now prove the theorem by induction on n. If $n = 1$ the result is trivial. We assume the result for all H with $\dim H < n$ and all possible choices for Σ. We now prove it for n.

For each i let m_i denote the number of j such that a_j is a multiple of a_i. Let $m = \max\{m_i | i = 1, ..., 2n\}$. If we relabel we may assume that the first m of the a_i are equal to a_1 (we may have to multiply h_i by a scalar of norm 1). By 1. above we may assume that the next k of the a_i are equal to $\widehat{a}_1$ with $1 \leq k \leq m$ and if $i > m+k$ then a_i is not a multiple of either a_1 or $\widehat{a}_1$. This implies by 2. above that $\langle h_i | h_j \rangle = 0$ for $j > m+k$ and $i = 1, ..., m$. Also $\{h_1, ..., h_m\}$ is an orthonormal set. Thus $u_i \in V \otimes (\{h_1, ..., h_m\}^{\perp})$ for $i > m+k$. This implies that $V \otimes (\{h_1, ..., h_m\}^{\perp})$ contains

$2n-(m+k)$ orthonormal elements. Since $\dim V\otimes(\{h_1,...,h_m\}^\perp)=2(n-m)$ this implies that $k=m$. We now rewrite the first $2m$ elements of the basis as

$$a_1\otimes b_1,...,a_1\otimes b_m,\widehat{a}_1\otimes c_1,...,\widehat{a}_1\otimes c_m.$$

If we apply observation 2. again we see that the elements h_i for $i>2m$ must be orthogonal to $\{b_1,...,b_m\}$ and to $\{c_1,...,c_m\}$. A dimension count says that they must span the orthogonal complements of both $\{b_1,...,b_m\}$ and $\{c_1,...,c_m\}$. But then $\{b_1,...,b_m\}$ and $\{c_1,...,c_m\}$ must span the same space, $U\subset H$. We have therefore shown that $\{u_i\}_{i>2m}$ is an orthonormal basis of $V\otimes U^\perp$. We may thus apply the inductive hypothesis to $U^\perp$ and $\Sigma\cap U^\perp$. This completes the inductive step and hence the proof.

If W is a Hilbert space and if Ξ is a subset of $S(W)$ that is invariant under multiplication by scalars of absolute value 1 then a function $f:\Xi\to\mathbb{C}$ is said to be a Ξ-frame function of weight $w=w_f$ if whenever $\{u_i\}$ is an orthonormal basis of W with $u_i\in\Xi$ (i.e. $\{u_i\}$ is a Ξ-frame) we have $\sum_i f(u_i)=w$. We note

3. Let f be a Ξ-frame function. If $\{u_i\}$ is a Ξ-frame for W and if F is a subset of $\{u_i\}$ then $f_{|F^\perp\cap\Xi}$ is a $F^\perp\cap\Xi$-frame function of weight $w_f-\sum_{u_i\in F}f(u_i)$.

This is pretty obvious. Let $\{v_j\}$ be a $\Xi\cap F^\perp$-frame for $F^\perp$. Then $\{\nu_j\}\cup F$ is a Ξ-frame for W.

Proposition 7. *Let V be a two dimensional Hilbert space and let H be an n-dimensional Hilbert space. Let $\Sigma\subset S(H)$ be as in the rest of this section and let $g:S(V)\to\mathbb{C}$ and $h:\Sigma\to\mathbb{C}$ be respectively a frame function and a Σ-frame function. Then if $f(v\otimes w)=g(v)h(w)$ for $v\in S(H)$ and $w\in\Sigma$ then f is an $S(V)\otimes\Sigma$-frame function of weight w_gw_h.*

Proof. Let $\{u_i\}$ be an $S(V)\otimes\Sigma$-frame. Then Theorem 5 implies that we may assume that there is partition $n_1\geq n_2\geq...\geq n_r>0$ of n and elements a_i,b_{ij} and c_{ij} as in the statement so that

$$\{u_i\}=\bigcup_{i=1}^{r}\left(\{a_i\otimes b_{ij}|j=1,...,n_i\}\cup\{\widehat{a}_i\otimes c_{ij}|j=1,...,n_i\}\right).$$

Thus

$$\sum_i f(u_i)=\sum_i g(a_i)\sum_{j=1}^{n_i}h(b_{ij})+\sum_i g(\widehat{a}_i)\sum_{j=1}^{n_i}h(c_{ij}).$$

Observation 3. above implies that for each i we have $\sum_{j=1}^{n_i}h(b_{ij})=\sum_{j=1}^{n_i}h(c_{ij})$. Now $g(a_i)+g(\widehat{a}_i)=w_g$. Hence since $\{b_{ij}\}$ is a Σ-frame the result follows.

Theorem 3 is an immediate consequence of the above proposition.

4. Entangled subspaces.

Let $H_1, ..., H_n$ be finite dimensional Hilbert spaces and set $H = H_1 \otimes H_2 \otimes \cdots \otimes H_n$. If $V \subset H$ is a subspace than we will say then V is *entangled* if whenever $v \in V$ and $v \neq 0$ then v is entangled (i.e. v cannot be written in the form $v = h_1 \otimes h_2 \otimes \cdots \otimes h_n$ for any choice of $h_i \in H_i$). The purpose of this section is to give a proof of Theorem 4 using basic algebraic geometry. That is, we will prove that

$$\dim V \leq \dim(H_1) \cdots \dim(H_n) - \sum(\dim H_i - 1) - 1$$

and that this estimate is best possible. The reader should consult [Hartshorne] for the algebraic geometry used in the proof of this result.

Let $L = \{\lambda \in H^* | \, \lambda(V) = 0\}$ (H^* the complex dual space of H). Let $X = \{h_1 \otimes \cdots \otimes h_n \, | \, h_i \in H_i\}$. We consider the map $\Phi : H_1 \times ... \times H_n \to X$ given by $\Phi(h_1, ..., h_n) = h_1 \otimes \cdots \otimes h_n$. Then Φ is a surjective polynomial mapping. If we denote by $\overline{\Phi}$ the corresponding mapping of projective spaces we have $\overline{\Phi} : P(H_1) \times ... \times P(H_n) \to P(H)$. General theory implies that the image of $\overline{\Phi}$ is Zariski closed in $P(H)$. Since X is clearly the cone on that image we see that X is Zariski closed and irreducible. Also the map $\overline{\Phi}$ is injective so the dimension over $\mathbb{C}$ of its image is $\sum(\dim H_i - 1)$. Thus the dimension over $\mathbb{C}$ of X is $d = \sum(\dim H_i - 1) + 1$.

Since V is entangled $X \cap V = \{0\}$. This implies that $\{x \in X \, || \lambda(x) = 0, \lambda \in L\} = \{0\}$. Thus $\dim L \geq \dim X = d$. Hence $\dim V = \dim H - \dim L \leq \dim H - d$. This is the asserted upper bound. The fact that this upper bound is best possible follows from the Noether normalization theorem which implies that there exist $\lambda_1, ..., \lambda_d \in H^*$ such that $\{x \in X \, || \lambda_i(x) = 0 \text{ for all } i\} = \{0\}$ (i.e. a linear system of parameters).

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