

Improved Two-Party and Multi-Party Purification Protocols

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ABSTRACT. We present an improved protocol for entanglement purification of bipartite mixed states using several states at a time rather than two at a time as in the traditional recurrence method. We also present a generalization of the hashing method to n -partite cat states, which achieves a finite yield of pure cat states for any desired fidelity. Our results are compared to previous protocols.

1. Introduction

Entanglement is a fundamental resource in quantum information. It can be used for secure quantum cryptography [1] and is an essential part of known algorithms for quantum computation [2, 3] (strangely, it is not known that all quantum algorithms which outperform their classical counterparts *require* entanglement. See [4] for a situation in which quantum states display a form of nonlocality, but which involves no entanglement).

Early studies of entanglement purification [5, 6, 7] focused mainly on bipartite entanglement, attempting to distill pure EPR pairs [8] from bipartite mixed states. More recently, Murao, Plenio, Popescu, Vedral and Knight [9] have studied the generalization of such schemes to distilling three-party (GHZ [10]) and multi-party states of the form (sometimes called “cat” states [11])

$$(1) \quad |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\dots 0\rangle + |11\dots 1\rangle)$$

from three-party and multi-party entangled mixed states. However, they do not study generalizations of the hashing method of [6]. The hashing scheme has the major advantage over the recurrence style scheme of [9] that it achieves a finite yield of pure cat states for any arbitrarily high fidelity, whereas the yield for any recurrence method goes to zero.

In this paper we study an improved purification protocol for two parties, and a generalization of hashing to multiple parties.

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First, we define some notation: All of our studies will apply to entangled mixed qubit states of N parties (conventionally known as Alice, Bob, etc.), diagonal in the following basis:

$$(2) \quad |\psi_{p,i_1 i_2 \dots i_{N-1}}\rangle = \frac{|0i_1 i_2 \dots i_{N-1}\rangle + (-1)^p |1\bar{i}_1 \bar{i}_2 \dots \bar{i}_{N-1}\rangle}{\sqrt{2}}$$

where p and the i 's are zero or one, and a bar over a bit value indicates its logical negation. This gives 2^N orthogonal states.

These states correspond to the simultaneous eigenvectors of the following operators (There are N operators in all, one special one of the X form, and $N - 1$ involving Z and I):

$$(3) \quad \begin{aligned} S_0 &= X \otimes X \otimes X \otimes X \dots X \\ S_1 &= Z \otimes Z \otimes I \otimes I \dots I \\ S_2 &= Z \otimes I \otimes Z \otimes I \dots I \\ S_3 &= Z \otimes I \otimes I \otimes Z \dots I \\ &\vdots \\ &\vdots \\ S_{N-1} &= Z \otimes I \otimes I \otimes I \dots Z \end{aligned}$$

The X , Z , and I operators, along with the Y operator which we don't use here, are members of the Pauli group (for more details, see [12]). The p from Equation (2) corresponds to whether a state is a $+1$ or -1 eigenvector of S_0 ($p = 0$ for a $+1$ eigenvector and $p = 1$ for a -1 eigenvector). This is called the "phase" bit of the cat state. The i_j s correspond to whether a state is a $+1$ or -1 eigenvector of S_j for $j = 1, \dots, (N - 1)$, which we call the amplitude bits. Thus, the following is the set of generators of the stabilizer group of $|\psi_{p,i_1 i_2 \dots i_{N-1}}\rangle$:

$$(4) \quad \{(-1)^p S_0, (-1)^{i_1} S_1, \dots, (-1)^{i_{N-1}} S_{N-1}\}$$

It is important to realize at this point that since these operators are all tensor products of operators on the subsystems $1 \dots N$ each with eigenvalue ± 1 , they can be measured using only local quantum operations plus classical communication. If an unknown one of the ψ 's is shared among N parties and they wish to determine the eigenvalue corresponding to one of the S_i , each party just measures his or her operator and reports the result to everyone else. The eigenvalue of the whole operator is the product of their individual results. Furthermore, since Z and I commute, it is possible to measure the eigenvalues of *all* the $S_{i>0}$. On the other hand, X and Z do not commute and therefore if S_0 is measured, none of the $S_{i>0}$ can be measured (a random result would occur) and similarly if any of the $S_{i>0}$ are measured the result of an S_0 measurement will be randomized. In other words, the parties can locally measure either all the amplitude bits or the phase bit for an unknown cat state.

The other tool we will need is the multilateral quantum XOR gate, in which each party's bits are XORed together in a quantum-coherent way (see Fig. 1). Following Gottesman [12] we can work out how a tensor product of two cat states behaves under the multilateral XOR. The generators of the stabilizer group behave

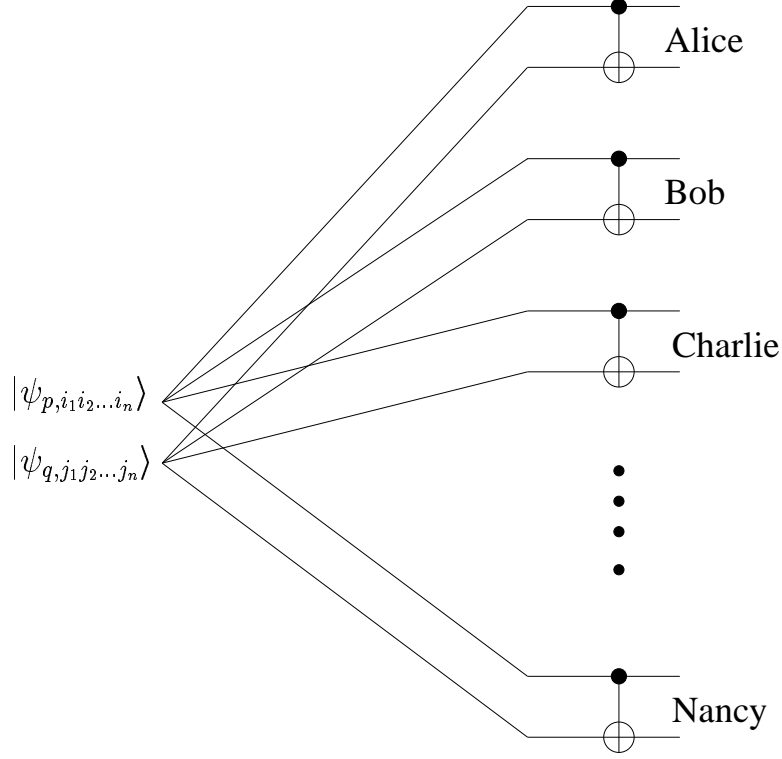


FIGURE 1. The multi-party XOR.

as follows under the quantum XOR operation:

$$(5) \quad \begin{aligned} X \otimes I &\rightarrow X \otimes X \\ I \otimes X &\rightarrow I \otimes X \\ Z \otimes I &\rightarrow Z \otimes I \\ I \otimes Z &\rightarrow Z \otimes Z \end{aligned}$$

We work out here the case of three parties, the generalization to n -partite cat states will be apparent. Given $|\psi_{p,i_1 i_2}\rangle$ and $|\psi_{q,j_1 j_2}\rangle$ with stabilizers as in Eq. (3) the generators of the stabilizers of the tensor product of these are given by:

$$(6) \quad \{(-1)^p XXXIII, (-1)^{i_1} ZZIIII, (-1)^{i_2} ZIZIII, \\ (-1)^q IIIXXX, (-1)^{j_1} IIIZZI, (-1)^{j_2} IIIZIZ\}$$

(We have omitted the \otimes symbol for brevity.) Now, applying the rule for the XOR operation (5) to corresponding operators (the first and fourth positions correspond to the first party's piece of $|\psi_{p,i_1 i_2}\rangle$ and $|\psi_{q,j_1 j_2}\rangle$ respectively, etc.) we get:

$$(7) \quad \{(-1)^p XXXXXX, (-1)^{i_1} ZZIIII, (-1)^{i_2} ZIZIII, \\ (-1)^q IIIXXX, (-1)^{j_1} ZZIZZI, (-1)^{j_2} ZIZZIZ\}$$

We can easily find another set of generators for the same stabilizer group which is again the tensor form of (6):

$$(8) \quad \left\{ \begin{array}{lll} (-1)^{p+q} XXXIII, & (-1)^{i_1} ZZIIII, & (-1)^{i_2} ZIZIII, \\ (-1)^q IIIXXX, & (-1)^{i_1+j_1} IIIZZI, & (-1)^{i_2+j_2} IIIZIZ \end{array} \right\}$$

This is simply the set of generators corresponding to $|\psi_{p \oplus q, i_1 i_2}\rangle \otimes |\psi_{q, i_1 \oplus j_1 \ i_2 \oplus j_2}\rangle$. What has happened is that the phase bits have been XORed together with the result put into the phase bit of the source state and the amplitudes are each XORed together and stored in the target state's amplitude bits. This suggests that the action of the multilateral quantum XOR gate (MXOR) can be characterized by its action on the purely classical representation of states as a set of bits, $(p, i_1, i_2, \dots, i_{N-1})$:

$$(9) \quad \text{MXOR}[(p, i_1, i_2, \dots, i_{N-1}), (q, j_1, j_2, \dots, j_{N-1})] = \\ (p \oplus q, i_1, i_2, \dots, i_{N-1}), (q, i_1 \oplus j_1, i_2 \oplus j_2, \dots, i_{N-1} \oplus j_{N-1})$$

Due to its linearity quantum mechanics allows us to think of mixed states as if they are really one of the pure states in the mixture but that we are simply lacking the knowledge of which one (if the states in the mixture come with unequal probability, we are not completely lacking knowledge of which state is in the mixture, but we only know the probabilities, not which state we actually have). Since all the cat states (2) are interconvertible by local operations [13] if we had a mixture of cat states and could determine which one we actually had, we would be able to convert it to a $|\Phi^+\rangle$ and would have purified the mixture.

Putting everything we have said up to now together lets us find purification schemes that are essentially classical; only the rules of what we can do are given by quantum mechanics:

- Mixed states diagonal in the cat basis can be thought of as being simply unknown members of the set of cat states.
- The cat states (2) are all interconvertible by local operations, so determining which cat state one has is sufficient to have purified it.
- Either the p or all the i 's of an unknown cat state may be measured by local operations plus classical communication.
- The multilateral XOR operation operates classically on the ps and i 's of pairs of cat states according to Eq. (10).

Our purification schemes will thus work by treating a set of many mixed states (which are diagonal in the cat basis) as a set of unknown cat states, and attempting to determine the unknown states, discarding them if we cannot.

We are now prepared to analyze the efficiencies of various entanglement purification protocols applied to mixed states. In particular, we concentrate on the generalization of the Werner state [14]:

$$(10) \quad \rho_W = \alpha |\Phi^+\rangle\langle\Phi^+| + \frac{1-\alpha}{2^N} \mathbb{1}, \quad 0 \leq \alpha \leq 1$$

The fidelity of ρ_W relative to the desired pure state $|\Phi^+\rangle$ is $F = \langle\Phi^+|\rho_W|\Phi^+\rangle = \alpha + \frac{1-\alpha}{2^N}$. We rewrite ρ_W in the cat basis (2) as

$$(11) \quad \rho_W = \left(\alpha + \frac{1-\alpha}{2^N} \right) |\psi_{0,00\dots 0}\rangle\langle\psi_{0,00\dots 0}| \\ + \frac{1-\alpha}{2^N} \sum_{p, i_1 i_2 \dots i_{N-1} \neq 0, 00\dots 0} |\psi_{p, i_1 i_2 \dots i_{N-1}}\rangle\langle\psi_{p, i_1 i_2 \dots i_{N-1}}|.$$

Thus, the Werner state is diagonal in the cat basis and we can think of it as really being one of the cat states. We write the unknown cat states as N unknown strings of bits: $b_0, b_1, b_2, \dots, b_{N-1}$, where b_0 is formed by concatenating the (unknown) phase bits of all the cat states, and the b_j for $j > 0$ are formed by concatenating the j th amplitude bits. Together the b_j make up the total bitstring B .

2. Bipartite Protocol

The case of two parties has been studied [5, 6, 7]. The protocols can distill pure entanglement from any Werner state with fidelity $F > 1/2$. The recurrence methods that work on Werner states near $F = 1/2$ involve local quantum operations on two mixed states at a time. For high fidelities, the best known strategy (the *hashing* method) obtains high yields in the limit of arbitrarily large numbers of states. It seemed that operations on an intermediate number of mixed states might give better yield for intermediate fidelities, and this turns out to be the case.

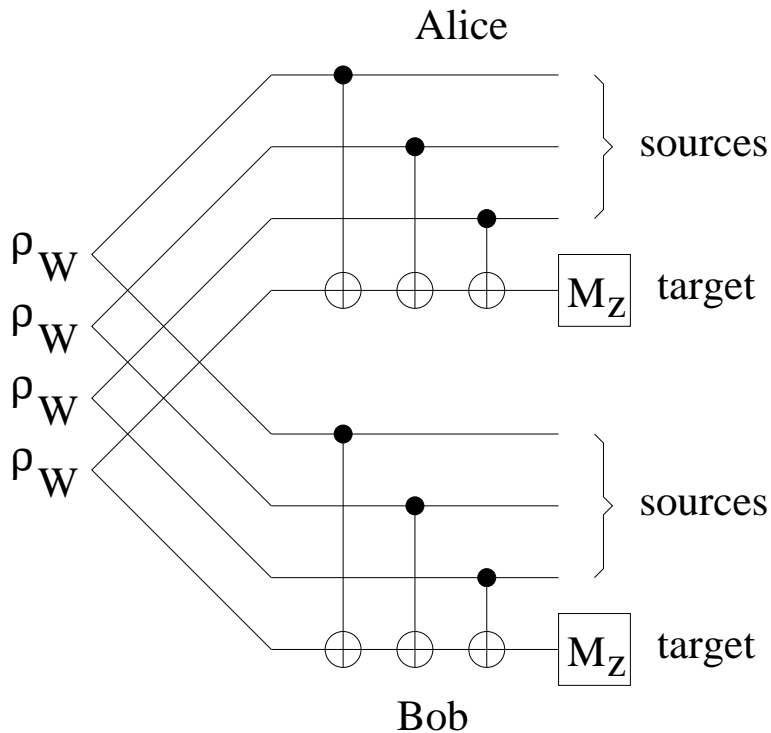


FIGURE 2. The sequence of XOR gates and final measurement in the z basis for our bipartite strategy for $m = 4$.

Our new strategy is to choose a block size m and to take $m - 1$ Werner states and do a bipartite XOR between each one and an m th Werner state, and then to measure the amplitude bits of that target state. The sequence of XORs is illustrated in Fig. 2 for the $m = 4$ case. This is a natural generalization of the recurrence method whose single step is just this method for $m = 2$. If any amplitude bit is nonzero, the two measurements disagree, the source states are discarded. If all amplitude bits are zero the states are said to have “passed” and the hashing method

is performed on the $m - 1$ source states along with other source states that passed. The advantage over the $m = 2$ recurrence is that fewer than half the mixed states are used up inherently just by being measured targets.

In [6] the hashing method was used only on states whose mixture probabilities were *independent*. We note that hashing is a quite general method for extracting entropy from strings of bits, even if there are correlations among the bits. One merely needs to take as many hash bits as there is entropy in the bitstring. Thus, the yield of our method is:

$$(12) \quad p_{\text{pass}} \frac{m-1}{m} \left(1 - \frac{H(\text{passed source states})}{m-1} \right)$$

The calculation of the entropy of a block of passed states and of p_{pass} is straightforward. One simply keeps track of the probability of each possible string B (of $2mN$ bits corresponding to a block of m Werner states) given the probabilities in the Werner mixture (12), applies the MXOR rule (10) to B to yield a B' and groups the like B' 's which have passed to yield a final distribution P_i (of the $2(m-1)N$ bits corresponding $m-1$ states left after the MXOR operation). We then have $p_{\text{pass}} = \sum_i P_i$ and the normalized distribution $P'_i = P_i/p_{\text{pass}}$ and the entropy $H(\text{passed source states})$ given by $-\sum_i P'_i \log_2 P'_i$.

Figure 3 compares the yield for our new method for various values of m with the previous recurrence continued by hashing protocol. We have not found a simple way to analyze what happens if our multi-bit step is iterated, rather than passing on immediately to hashing. For the $m = 2$ recurrence only one passed source state remains and it is identical in all respects to every other passed source state. For $m > 2$ there are multiple correlated passed states and it is not clear just how to treat them. For instance, at $m = 3$ there are two passed states from each operation and there is no way to combine the 4 passed states from two operations into another $m = 3$ step.

3. Multipartite Hashing

In [9] multi-party recurrence methods are studied, but not multi-party hashing which is needed to achieve finite yields. Here we present a multi-party hashing method.

In the case of two parties it is known [6] how to extract the parity of any random subset of all the bits in B . For more than two parties it is not known now to do this. Instead, we can choose to extract any random subset parity on either the parity bitstring b_0 or on all the amplitude bitstrings b_j , $j > 0$ in parallel. This follows immediately from Eq. (10): This is done multilaterally XORing together all the states in the desired subset choosing one of them to be the target. See Fig. 4. Depending on the direction of the XOR gates either the phase bits or all the amplitude bits accumulate in the target state, which can then be measured.

Our multilateral hashing protocol will be to choose a large block size m and then to extract $m \max_{j>0} [H(b_j)]$ random subsets of each amplitude bitstring in parallel (as shown in Figure 4a), where $H(b_j)$ is the entropy per bit in string b_j . This is sufficient to determine all the bits of all the $b_{j>0}$ as it is just doing the same random hash on each bitstring. Even though the random hashes are all the same, since they are uncorrelated with the bitstrings being determined this many hash bits will be enough to determine all bits of the $b_{j>0}$ [15]. This procedure actually extracts too much information (and thereby uses up too many states as measured

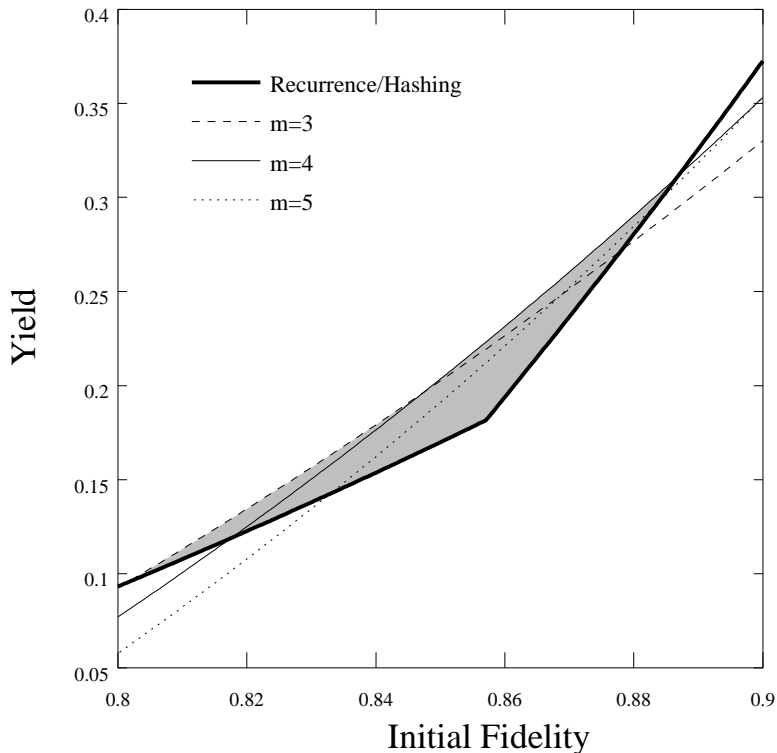


FIGURE 3. Yields of various bipartite purification protocols. The thick line is the recurrence continued by hashing method, the dashed line is our new method for $m = 3$, solid for $m = 4$ and dotted for $m = 5$. The shaded region is the region where our new method improves on the recurrence/hashing method. Note that yield for $m = 5$ is never the best, though for some fidelities it is better than the recurrence/hashing method. The $m = 6$ case (not shown) behaves similarly, while for $m > 6$ the new method is always worse than recurrence/hashing. The sharp “knee” visible in the recurrence/hashing line is the point above which recurrence is never used and hashing is done immediately.

targets), so perhaps a more efficient protocol exists, but this has the virtue of using only the multilateral XOR operation which maps cat states to cat states. After determining the amplitude bits, to find b_0 we use multilateral XORs arranged as in Fig. 4b, and find the hash of the string by measuring another $H(b_0)$ of the states. The yield of this hashing protocol D_h is given by

$$(13) \quad D_h = (1 - \max_{j>0} \{H(b_j)\}) - H(b_0)$$

For the case of Werner states all the b 's have the same entropy and Eq. (13) reduces to

$$(14) \quad D_W = 1 - 2H_2 \left(\frac{(1-f)2^{N-1}}{2^N - 1} \right)$$

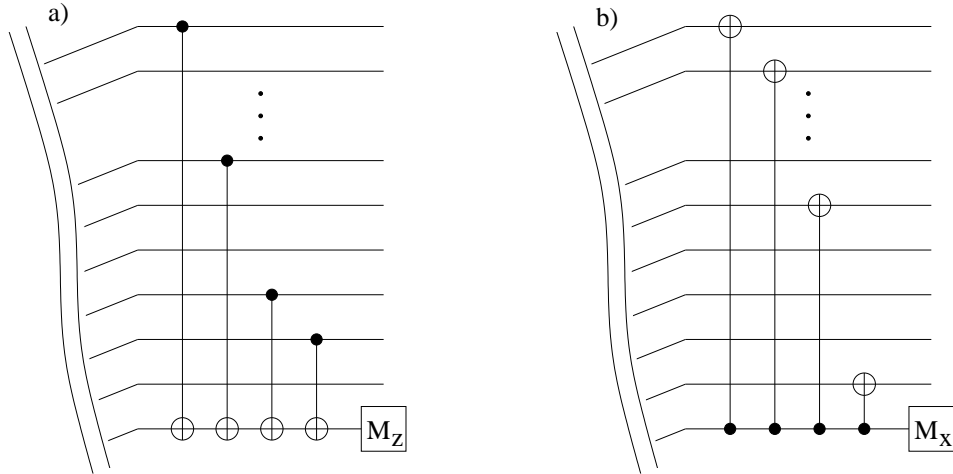


FIGURE 4. Multi-party hashing: These hashes are done on large blocks of bits (indicated by the vertical ellipsis) and are done multilaterally (only one party's operations are shown, the other $N - 1$ parties operations are identical).

a) Finding a random subset parity on all the $b_{j>0}$ in parallel. In this case the first, third, sixth and seventh states shown are XORed multilaterally into the last one which is then measured to determine the eigenvalue of the Z operator.

b) Finding a random subset parity on b_0 . In this instance the parity of the first, second, fourth and eighth states shown are XORed with the last one, which is then measured in the eigenbasis of the X operator. Note the reversal of the direction of the XOR gates with respect to a).

or in the limit as the number of parties goes to infinity

$$(15) \quad D_W^\infty = 1 - 2H_2\left(\frac{1-f}{2}\right).$$

where $H_2(x) = -x \log_2 x - (1-x) \log_2 (1-x)$. Eq. 14 is graphed for several values of N in Figure 5. By using the recurrence method and switching to hashing as soon as it gives better yield, one can obtain positive final yield to arbitrarily high fidelity for any initial fidelity for which the recurrence method of [9] improves the fidelity.

4. Conclusions and Comments

We have found improved bipartite recurrence protocols for the purification of entanglement from mixed quantum states. We have also demonstrated the first finite-yield method for purification of cat states in a multi-party setting. It is worthwhile to note that both of these new procedures were analyzed for mixed state diagonal in the cat basis, but that in fact they will work for *any* mixed state just as well, by considering the state's cat-basis diagonal elements. This is unlikely to be the optimally efficient strategy for non-diagonal states however. In

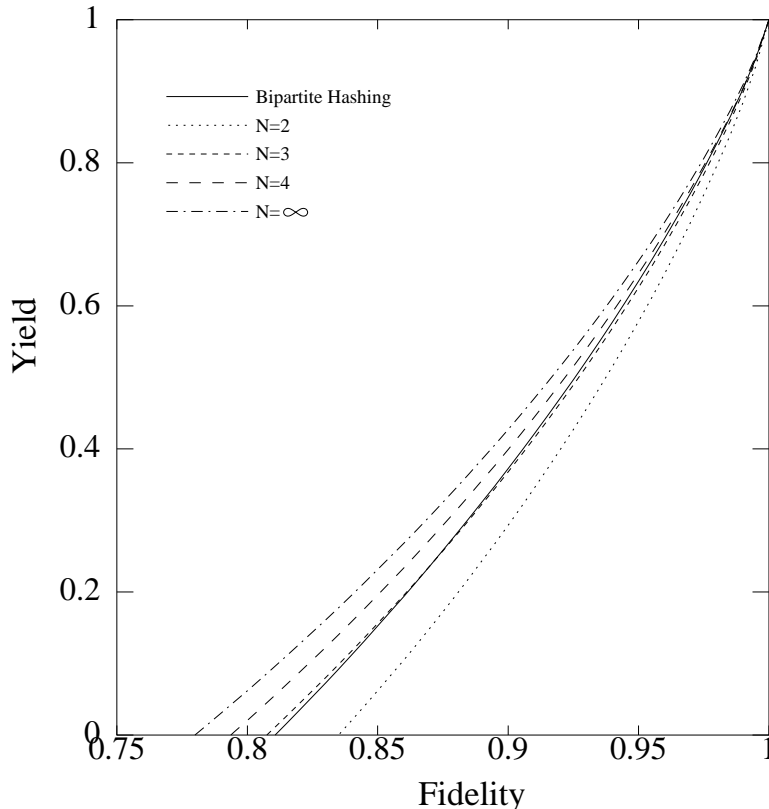


FIGURE 5. Yields for multipartite hashing for various numbers of parties. The solid line is the two-party hashing method of [6]. The dotted line is the corresponding $N = 2$ version of our new hashing method, which has a lower yield since it works on the amplitude and phase bits as separate hash strings even though for the bipartite case it is known how to extract their entropy together, which is more efficient. The lines consisting of dashes, longer dashes and dots with dashes are the $N = 3$, $N = 4$, and $N = \infty$ cases respectively.

[6] there is an example of a state for which the conventional bipartite recurrence and hashing cannot distill any pure entanglement, but which can nevertheless simply be distilled. One expects such examples to exist for our new methods as well (indeed, the example in [6] *is* an example for the bipartite case of new methods which will similarly fail to distill it.

For our bipartite protocol, while clearly not optimal it is not so bad to have passed over to hashing instead of recurring the protocol. Recurrence methods have vanishing yield if one desires arbitrarily high fidelity of the purified states, so hashing needs to be used eventually in any case. Additionally it would likely be best to produce a variable block size protocol that begins as the recurrence

method for low fidelity, switches to a larger block size at some higher fidelity and finally is continued by hashing. A calculation of the yield of such a method is cumbersome, and seemingly provides little insight. We hope that our having pointed out that block size $m > 2$ methods can improve over recurrence will stimulate further work in this area to develop a deeper understanding, rather than just a brute-force analysis. Much progress has been made on purification involving only one-way classical communication. Such protocols directly correspond to quantum error-correcting codes (cf. [6]) but recurrence protocols inherently involve two-way classical communication so all parties know which states to discard. So far little of the coding theory has been applied to this case. There does appear to be some relation between these two-way purification protocols and quantum error-detecting codes, and some progress is being made in this area [16].

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