QUANTUM INFORMATION THEORY

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Cerf/Adami

Einstein/Podolsky/Rosen

Von Neumann

Shannon

Bennett

Deutsch

Feynman

Peres

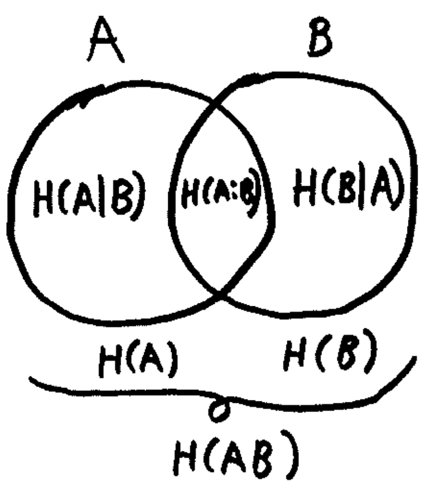
Wooters

. . .

Sundry Formulas

$$\begin{cases} H(AB) = H(A) + H(B|A) \\ \qquad\qquad = H(B) + H(A|B) \\ H(A:B) = H(A) + H(B) - H(AB) \end{cases}$$

$$\begin{cases} H(AB) = H(A|B) + H(A:B) + H(B|A) \\ H(A) = H(A|B) + H(A:B) \\ H(B) = H(B|A) + H(A:B) \end{cases}$$

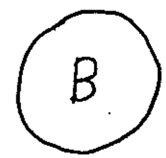


Two Coins

Coin A	0	1
Prob	1/2	1/2

$H(A) = 1$

Coin B	0	1
Prob	1/2	1/2



$H(B) = 1$

Consider 2 Sources A & B of Qubits

Source A	$\|0\rangle$	$\|1\rangle$
Prob	1/2	1/2

Source B	$\|0\rangle$	$\|1\rangle$
Prob	1/2	1/2

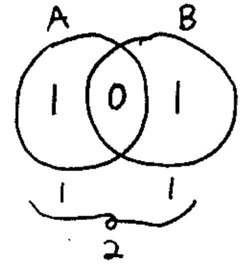
$$\therefore H(A) = 1 = H(B)$$

Source E	$\frac{\|0\rangle + \|1\rangle}{\sqrt{2}}$
Prob	1

$$H(E) = 0$$

Case I. (Classical) Two stochastically uncorrelated qubits

Source AB	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|11\rangle$
Probability	1/4	1/4	1/4	1/4

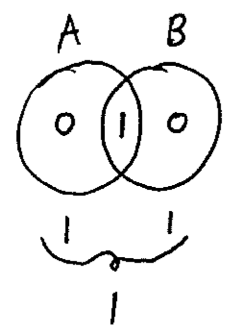


$H(AB) = 2$

Mixed Ensemble

Case II. (Classical) Two stochastically dependent qubits

Source AB	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|11\rangle$
Probability	1/2	0	0	1/2



$H(AB) = 1$

Mixed Ensemble

Mixed Ensemble

Unit Length Kets

Ket	$\vert\psi_1\rangle$	$\vert\psi_2\rangle$	$\cdots$	$\vert\psi_n\rangle$
Prob	p_1	p_2	$\cdots$	p_n

$$\sum_{k=1}^{n} p_k = 1$$

mixed state is represented by:

$$\rho = p_1|\psi_1\rangle\langle\psi_1| + p_2|\psi_2\rangle\langle\psi_2| + \cdots + p_n|\psi_n\rangle\langle\psi_n|$$

density op.

Density Operator of a Q.M. Ensemble

Example Consider the mixed ensemble

Qubit A	$\vert 0\rangle$	$\vert 1\rangle$
Prob	½	½

The density operator ρ of A is defined as

$$\rho_A = \tfrac{1}{2}|0\rangle\langle 0| + \tfrac{1}{2}|1\rangle\langle 1|$$

ρ_A "Lives" in $\mathrm{Hom}(\mathcal{H}, \mathcal{H})$

Given the density operator ρ_{AB} of AB, what is the density operator ρ_A of **A** if we ignore B?

$$\begin{tikzcd} AB \arrow[d, "\text{Ignore } B"] & \rho_{AB} \arrow[d, dashed, "?"] \\ A & \rho_A \end{tikzcd}$$

Answer: The partial trace

$$\boxed{\rho_A = \mathrm{Trace}_B(\rho_{AB})}$$

The Partial Trace

Consider Hilbert spaces $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}_A \otimes \mathcal{H}_B$. Let $d_A = \dim(\mathcal{H}_A)$ & $d_B = \dim(\mathcal{H}_B)$. Hence $d_A d_B = \dim(\mathcal{H}_A \otimes \mathcal{H}_B)$.

Consider $\rho_{AB} \in \mathrm{Hom}(\mathcal{H}_A \otimes \mathcal{H}_B, \mathcal{H}_A \otimes \mathcal{H}_B)$

Then

$$\begin{tikzcd} \mathrm{Hom}(\mathcal{H}_A \otimes \mathcal{H}_B, \mathcal{H}_A \otimes \mathcal{H}_B) \arrow[r, "\text{Basis } \{e_i \otimes f_j\}", "\Phi"'] \arrow[d, dashed, "\mathrm{Trace}_B"'] & \mathcal{M}_{d_1 d_2}(\mathbb{C}) \arrow[d, "\widetilde{\mathrm{Trace}}_B"] \\ \mathrm{Hom}(\mathcal{H}_A, \mathcal{H}_A) \arrow[r, "\tilde{\Phi}", "\text{Basis } \{e_i\}"'] & \mathcal{M}_{d_1}(\mathbb{C}) \end{tikzcd}$$

$$\rho_A = \mathrm{Trace}_B(\rho_{AB})$$

$$= \tilde{\Phi}_2^{-1}\ \widetilde{\mathrm{Trace}}_B\left[\Phi(\rho_{AB})\right]$$

Von Neumann Entropy(Cont.)

Entropy	$S(A) = -Trace\left(\rho_A \lg \rho_A\right)$	Uncertain of A
Joint Entropy	$S(AB) = -Trace\left(\rho_{AB} \lg \rho_{AB}\right)$	Joint uncertaint of A & B
Conditional Entropy	$S(A\|B) = -Trace\left(\rho_{AB} \lg \rho_{A\|B}\right)$	Uncertant of A given
Mutual Entropy	$S(A:B) = -Trace\left(\rho_{AB} \lg \rho_{A:B}\right)$	Uncertain shared by A & B

where

$$\begin{cases} \rho_{A|B} = \lim_{n\to\infty} \left[\rho_{AB}^{1/n} \left(1_A \otimes \rho_B\right)^{-1/n}\right] \approx \rho_{AB} \left(1_A \otimes \rho_B\right)^{-1} \\ \rho_{A:B} = \lim_{n\to\infty} \left[\left(\rho_A \otimes \rho_B\right)^{1/n} \rho_{AB}^{-1/n}\right] \approx \left(\rho_A \otimes \rho_B\right) \rho_{AB}^{-1} \end{cases}$$

Von Neumann Entropy is the Physical Entropy

Why?

- It is invariant under (i.e., preserved by) unitary transformations.

$$S(U\rho U^{-1}) = S(\rho)$$

- $S(A|B)$ & $S(A:B)$ are invariant under unitary transformations that preserve entanglement with respect to the decomposion $\mathcal{H}_A \otimes \mathcal{H}_B$, i.e., w.r.t. unitary transformations of the form

$$U_A \otimes U_B$$

Case III. 2 Entangled Qubits

$$|AB\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Pure Ensemble

No Uncertainty

$$\rho_{AB} = |AB\rangle\langle AB|$$
$$= \left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)$$

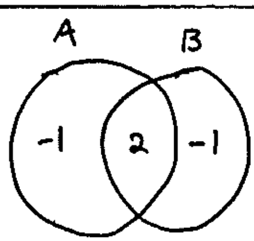
$\text{Trace}_B \downarrow$ Ignore **B**

$$\rho_A = \text{Trace}_B(\rho_{AB}) = \tfrac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$

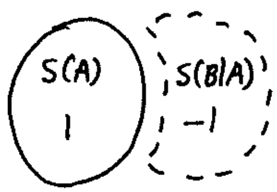
A	$\|0\rangle$	$\|1\rangle$
Prob	½	½

Mixed Ensemble

Uncertainty Introduced



$\xRightarrow{\text{Ignore } B}$

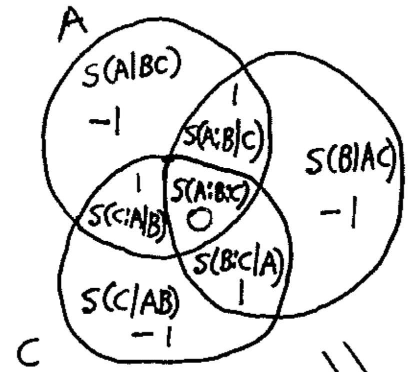


Total Entropy still zero !

Case IV 3 Entangled Qubits

$$|ABC\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

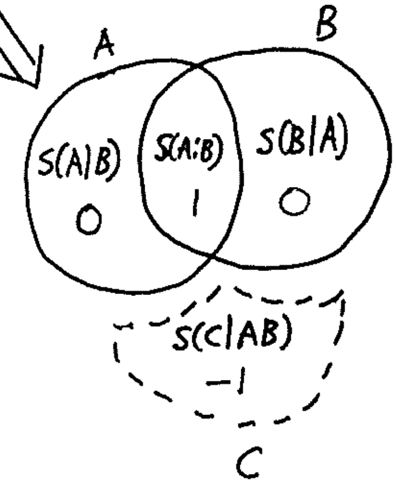
S(ABC) = 0
No Uncertainty !



Ignore C

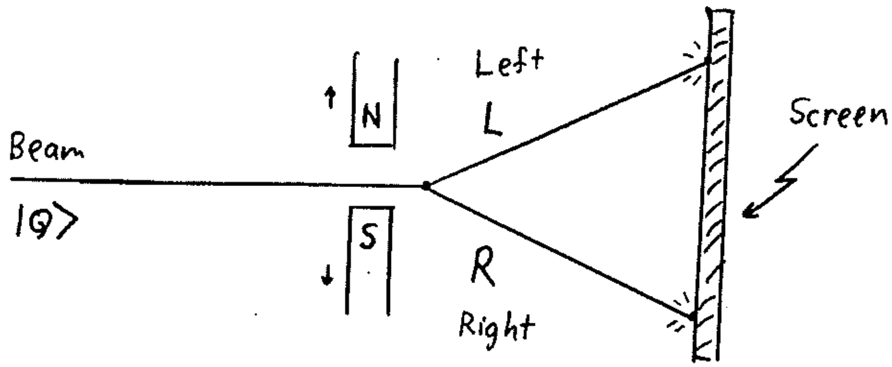
Trace_C

S(AB) = 1
Uncertainty !



Looks like Case II.

Stern - Gerlach Experiment



Init. Beam

$$|Q\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)$$

Apply Magn. Field

$$\Big\Downarrow\ U$$

Entangled
A = Spacial Loc.
Eigenstates $|L\rangle, |R\rangle$

$$|QA\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow, L\rangle + |\downarrow, R\rangle\right)$$

Interact w. Screen

$$\Big\Downarrow\ U'$$

Entangled
A' = Screen
Eigen states $|l\rangle, |r\rangle$

$$|QAA'\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow, L, l\rangle + |\downarrow, R, r\rangle\right)$$

Ignore Q

$$\Big\Downarrow$$

$$\therefore\ \rho_{AA'} = \frac{1}{2}\left(|L, l\rangle\langle l, L| + |R, r\rangle\langle r, R|\right)$$

System appears to be a mixture of 2 particles