Quantum Noise & Quantum Decoherence

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Two Ways of Representing Quantum States

Ket $|\psi\rangle$ vs. Density Operator ρ

Example. We have seen **pure ensembles**, i.e, pure states. For example,

Ket	$\|\psi\rangle$
Prob	1

Problem. Certain other types of quantum states are difficult to represent in terms of kets $|\psi\rangle$

If for example,

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

where

$$|a|^2 + |b|^2 = 1$$

then

$$\rho = (a|0\rangle + b|1\rangle)(a\langle 0 + b\langle 1\,)$$

$$= \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} a^* & b^* \end{pmatrix}$$

$$= \begin{pmatrix} |a|^2 & ab^* \\ ba^* & |b|^2 \end{pmatrix}$$

On the other hand,

$$\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$$

$$= \begin{pmatrix} \frac{7}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} \end{pmatrix}$$

is the mixed ensemble

Ket	$\|0\rangle$	$\|1\rangle$
Prob	$\frac{7}{8}$	$\frac{1}{8}$

Question. What happens when we ignore a component of a composite quantum system?

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We need one more tool, namely, the

Partial Trace

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Consider a quantum system Q_{PE} which is a composite of the environment Q_E and our principal quantum system Q_P. Then

$$\rho_{PE} = \sum_{r,s,t,u} \lambda_{rstu} |\psi_r^P\rangle|\psi_s^E\rangle\langle\psi_t^E|\langle\psi_u^P|$$

$$\Downarrow$$

Partial Trace $Tr_E \Downarrow$ Ignore Q_E

$$\Downarrow$$

$$\rho_P = Tr_E(\rho_{PE}) = \sum_{r,s,t,u} \lambda_{rstu} \langle\psi_t^E|\psi_s^E\rangle|\psi_r^P\rangle\langle\psi_u^P|$$

where we have performed the contraction

$$|\psi_s^E\rangle\langle\psi_t^E| \mapsto \langle\psi_t^E|\psi_s^E\rangle$$

to obtain ρ_P of Q_P.

We have "traced over the environment"

Hence by ignoring the environment, we have created uncertainty!

Pure	Ignore Q_E	Mixed
Ensemble	⇒⇒⇒⇒⇒⇒⇒	Ensemble
Q_{PE}	Tr_E	Q_P

We have created uncertainty!

Purification

Surprisingly enough, we can also do the reverse

Mixed Ensemble Q_P	Extend to Higher Dim Q. Sys. ⇒⇒⇒⇒⇒⇒⇒ ***Purification***	Pure Ensemble Q_{PE}

There are many different such extnsions which all produce through unitary evolution the same behavior of Q_P.

Evolution

Closed System

$$\rho \rightarrow \boxed{U} \rightarrow U\rho U^{\dagger}$$

Non-Closed System

$$\begin{array}{cccc} \rho & \rightarrow & \multirow{2}{*}{$\boxed{U}$} & \rightarrow \mathsf{E}(\rho) \\ \rho_E & \rightarrow & & \rightarrow \text{Ignore} \end{array}$$

Methods for dealing with non-unitary evolution

Method 1. *Operator Sum Representation*

$\exists$ ops. $\{E_a\}$ such that

$$\rho(t) = \mathsf{E}_t(\rho^{init}) = \sum_a E_a(\rho^{init})E_a^{\dagger}$$

Method 2. *Purification*

$$|e\rangle \otimes |\psi\rangle \overset{\text{Entangle}}{\longmapsto} \sum_s |e_s\rangle \otimes M_s|\psi\rangle$$

- $|e\rangle$: init. environment
- $|\psi\rangle$: init. state
- $|e_s\rangle$: Final Environment; not. nec. orthognal, not nec. normalized
- M_s: Error Pattern Unitary op.

Notation

Let $a = (a_1, a_2)$ and
$b = (b_1, b_2) \in \{00, 01, 10, 11\}$
Then all 16 of the 2 qubit error patterns can be uniquely written as

$$\{ X^a Z^b \mid a, b \in \{00, 01, 10, 11\} \}$$

where

$$X^a Z^b = X_1^{a_1} X_2^{a_2} Z_1^{b_1} Z_2^{b_2}$$

For example,

$$X^{(0,1)} Z^{(1,1)} = X_1^0 X_2^1 Z_1^1 Z_2^1 = I_1 X_2 Z_1 Z_2$$

$$= (I \otimes I)(I \otimes X)(Z \otimes I)(I \otimes Z)$$

$$= Z \otimes XZ = Z \otimes Y = Z_1 Y_2$$

Types of Error Patterns

$X^{(0,0)} Z^{(0,0)} = I \otimes I$	0-qubit Error Pattern
$X^{(1,0)} Z^{(1,0)} = Y \otimes I$	1-qubit Error Pattern
$X^{(1,1)} Z^{(1,0)} = Y \otimes X$	2-qubit Error Pattern

In general

$X^a Z^b$	$Wt(a \vee b)$-qubit Error Pattern

where '$\vee$' denotes bitwise logical 'OR', and where $Wt(a \vee b)$ denotes the Hamming weight of $a \vee b$.