Therefore,

$$ab = 0\ ,$$

either $a = 0$ and/or $b = 0$.

But a and b are arbitrary.

A Contradiction!

No Cloning Theorem

Definition. Let $\mathcal{H}$ be a Hilbert space. Then a **quantum replicator** consists of an auxiliary Hilbert space $\mathcal{H}_A$, a fixed state $\left|\psi_{\#}\right\rangle \in \mathcal{H}_A$ (called the **initial state of replicator**), and a unitary transformation

$$U : \mathcal{H}_A \otimes \mathcal{H} \oplus \mathcal{H} \longrightarrow \mathcal{H}_A \otimes \mathcal{H} \oplus \mathcal{H}$$

such that, for some fixed state $|blank\rangle \in \mathcal{H}$,

$$U\left|\psi_{\#}\right\rangle |a\rangle |blank\rangle = |\psi_a\rangle |a\rangle |a\rangle ,$$

for all states $|a\rangle \in \mathcal{H}$, where $|\psi_a\rangle \in \mathcal{H}_A$ (called the **replicator state after replication** of $|a\rangle$) depends on $|a\rangle$.

Key Idea

Key Idea. Cloning is inherently non-linear, whereas, quantum mechanics is inherently linear. **Ergo**, quantum replicators do not exist.

Quantum Mechanics from the Two Perspectives

Kets $\lvert\psi\rangle$	**Density Ops.** ρ	
$i\hbar\frac{\partial\lvert\psi\rangle}{\partial t} = H\lvert\psi\rangle$	$i\hbar\frac{\partial\rho}{\partial t} = [H,\rho]$	Schröd. Eq.
$\lvert\psi_0\rangle \mapsto U\lvert\psi_0\rangle$	$\rho_0 \mapsto U\rho_0 U^\dagger$	Unitary Evol.
$\langle A\rangle = \langle\psi \mid A \mid \psi\rangle$	$\langle A\rangle = Trace(A\rho)$	Observ.

- We now have a more powerful way of representing quantum states

- Density operators are absolutely crucial when discussing and dealing with quantum noise.

Example. Consider the following state for which we have incomplete knowledge, called a **mixed ensemble**:

Ket	$\|\psi_1\rangle$	$\|\psi_2\rangle$	$\cdots$	$\|\psi_k\rangle$	$\leftarrow \left\{ \begin{array}{l} \text{all unit lngth} \\ \text{\& not nec. } \perp \end{array} \right.$
Prob	p_1	p_2	$\cdots$	p_k	

where

$$p_1 + p_2 + \ldots + p_k = 1$$

We have incomplete knowledge of this state

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Johnny von Neumann suggested that we use the following operator to represent this state:

$$\rho = p_1|\psi_1\rangle\langle\psi_1| + p_2|\psi_2\rangle\langle\psi_2| + \ldots + p_k|\psi_k\rangle\langle\psi_k|$$

ρ is called a **density operator**. It is a Hermitian positive definite operator of trace 1.

For our pure ensemble:

Ket	$\|\psi\rangle$
Prob	1

$$\boxed{\rho = 1 \cdot |\psi\rangle\langle\psi|}$$

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We begin by noting that U_f when restricted to the orthonormal basis $\{|y, x_1 x_0\rangle\}$ is a classical permutation, i.e.,

$$\boxed{|y, x_1 x_0\rangle \longmapsto |y \oplus f(x_0, x_1), x_1 x_0\rangle}$$

0	$\|000\rangle \longmapsto \|000\rangle$	0
1	$\|001\rangle \longmapsto \|101\rangle$	5
2	$\|010\rangle \longmapsto \|110\rangle$	6
3	$\|011\rangle \longmapsto \|111\rangle$	7
4	$\|100\rangle \longmapsto \|100\rangle$	4
5	$\|101\rangle \longmapsto \|001\rangle$	1
6	$\|110\rangle \longmapsto \|010\rangle$	2
7	$\|111\rangle \longmapsto \|011\rangle$	3

This is the permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 5 & 6 & 7 & 4 & 1 & 2 & 3 \end{pmatrix},$$

which when written as the product of disjoint cycles becomes

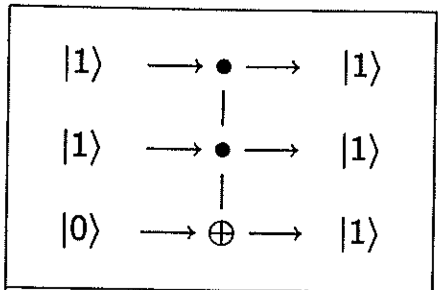
$$(15)\,(26)\,(37)$$

It follows that the the corresponding permutation matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which when interpreted as a unitary transformation becomes the unitary transformation U_f.

Moreover,

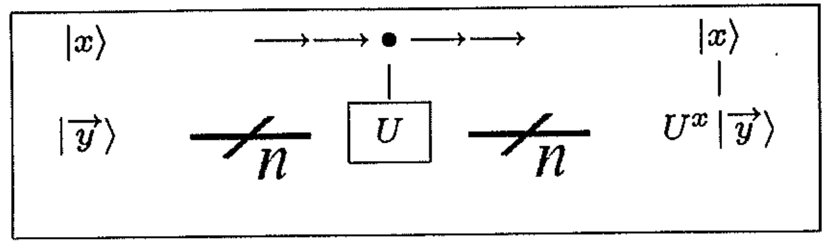


$$|011\rangle \longmapsto |111\rangle$$

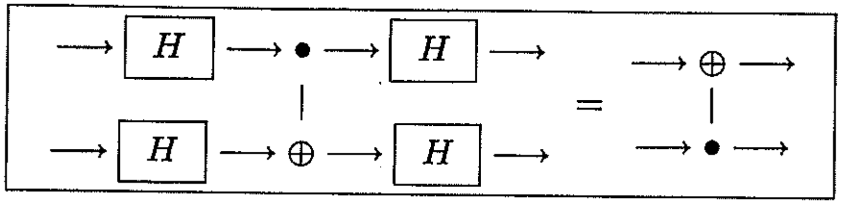
denotes

$$\begin{array}{c} \\ 000 = 0 \\ 001 = 1 \\ 010 = 2 \\ 011 = 3 \\ 100 = 4 \\ 101 = 5 \\ 110 = 6 \\ 111 = 7 \end{array} \quad \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Example. Let U be an n-qubit unitary transformation, Then a controlled-U gate is



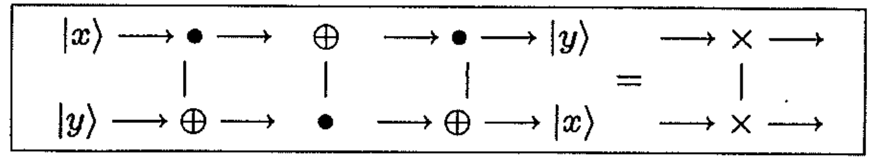
Observation. $CNOT$ and $CNOT'$ can be transformed into one another by the coodinate transformation $H \otimes H$, i.e.,



In other words, if we "tilt our heads by 45 degrees," the target and control switch.

Caveat Emptor. Wiring diagrams, like matrices, are basis dependent.

Example. The $SWAP$ gate



The n-Qubit Hadamard Transform

For the n-qubit Hadamard transform

$$H^{\otimes n} = \underbrace{H \otimes \cdots \otimes H}_{n},$$

we have

$$H^{\otimes n}|x\rangle = H|x_0\rangle \otimes \cdots \otimes H|x_{n-1}\rangle = \bigotimes_{j=0}^{n-1} H|x_0\rangle$$

$$= \bigotimes_{j=0}^{n-1} \frac{|0\rangle + (-1)^{x_j}|1\rangle}{\sqrt{2}} = \bigotimes_{j=0}^{n-1} \left(\sum_{z_j=0}^{1} \frac{(-1)^{x_j \cdot z_j}|z_j\rangle}{\sqrt{2}} \right)$$

$$= \frac{1}{2^{n/2}} \sum_{z_0=0}^{1} \cdots \sum_{z_{n-1}=0}^{1} (-1)^{\sum_{j=0}^{n-1} x_j z_j} |z_0\rangle \ldots |z_{n-1}$$

$$= \frac{1}{2^{n/2}} \sum_{z=0}^{2^n-1} (-1)^{x \cdot z} |z\rangle$$

Please note that

$$|x\rangle = |x_0\rangle \cdots |x_{n-1}\rangle \text{ and } |z\rangle = |z_0\rangle \ldots |z_{n-1}\rangle,$$

where the labels x and z within the kets denote respectively the integers

$$x = \sum_{j=0}^{n-1} x_j 2^j \text{ and } z = \sum_{j=0}^{n-1} z_j 2^j,$$

and that

$$x \cdot z$$

denotes the inner product of two binary n-tuples, i.e.,

$$(x_0, \ldots, x_{n-1}) \cdot (z_0, \ldots, z_{n-1}) = x_0 z_0 + \ldots x_{n-1} z_{n-1}$$

Wiring Diagrams

of

Qubit Devices

Wiring Diagrams

Wiring diagrams are a convenient way to describe unitary transformations.

Reason. While the size of the matrix representation grows exponentially with the dimension d of the quantum system, the complexity of the wiring diagram grows only linearly with d.

Example. The **NOT** gate, a.k.a., the **Bit Flip** gate is

$$\boxed{\sigma_1 = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

with corresponding wiring diagram

$$\boxed{|x\rangle \longrightarrow \boxed{X} \longrightarrow |x \oplus 1\rangle}$$

From Observables to Unitary Transfs.

Let $\mathcal{O}$ be an observable. Then

$$U = e^{i\mathcal{O}} = \sum_{k=0}^{\infty} \frac{(i\mathcal{O})^k}{k!}$$

is a unitary transformation.

Example.

Let

$$\mathcal{O} = \theta\sigma_2 = \theta \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Then

$$\begin{aligned} U &= e^{i\theta\sigma_2} = \cos(\theta)\,\sigma_0 + i\sin(\theta)\,\sigma_2 \\ &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \end{aligned}$$

which is a rotation θ, where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In general, if

$$\begin{aligned} \mathcal{O} &= \theta\,(a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3) \\ &= \theta\,(a_1, a_2, a_3)\cdot(\sigma_1, \sigma_2, \sigma_3) \\ &= \theta\,\overrightarrow{a}\cdot\overrightarrow{\sigma} \end{aligned}$$

where

$$\overrightarrow{a} = (a_1, a_2, a_3)$$

is a unit length vector in $\mathbb{R}^3$, and where

$$\overrightarrow{\sigma} = (\sigma_1, \sigma_2, \sigma_3),$$

Then

$$e^{i\theta\overrightarrow{a}\cdot\overrightarrow{\sigma}} = \cos(\theta)\,\sigma_0 + \sin(\theta)\,(\overrightarrow{a}\cdot\overrightarrow{\sigma})$$

which is a rotation about the axis $\overrightarrow{a}$ by the angle θ.

For this reason, $\overrightarrow{a}\cdot\overrightarrow{\sigma}$ is an **infinitesimal rotation**.