On the Bisection Width of the Transposition Network

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Abstract

The transposition network T_n of order $n!$ is the Cayley graph of the symmetric group S_n with generators the set of all transpositions in S_n. Finding the bisection width of the transposition network is an open question posed by F. T. Leighton. We resolve this question for n even, by showing that the bisection width of the transposition network T_n is equal to $nn!/4$. When $n \geq 2$ is odd, we show that the bisection width of the transposition network T_n is $(1 + o(1))nn!/4$. In doing so we determine the second smallest eigenvalue of the adjacency matrix of T_n.

Further, given a Cayley graph of a finite group G, with m conjugacy classes and with a set of generators closed under taking conjugates and not containing the identity of G, we show how to construct an $m \times m$ integer matrix Q, from the conjugacy classes of G, such that the set of eigenvalues of Q is equal to the set of eigenvalues of the adjacency matrix A of the given Cayley graph. Hence, when the order of G is large compared to the number of its conjugacy classes, a faster method for computing the eigenvalues of A is provided.

Keywords: parallel architectures, symmetric group, transpositions, bisection width, eigenvalues of graphs, divisors of graphs, algebraic connectivity of graphs, Cayley graphs.

1 Introduction

Akers and Krishnamurthy [1] and Carlsson *et al* [4] introduce a group–theoretic model, called the *Cayley graph model*, for designing and analyzing interconnection networks of parallel

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machines. Examples of such interconnection networks include hypercubes, cube–connected–cycles, and other hypercubic networks. Akers and Krishnamurthy [1] introduce the star network, which is the Cayley graph of the symmetric group S_n with generators a certain set of $n-1$ transpositions [1]. They show that the star network has better performance than the hypercube, where performance is measured, among other things, by the network's degree, diameter, connectivity, fault tolerance, and routing algorithms. The transposition network is another example of such an interconnection network.

The bisection width of an (undirected) graph is the minimum number of edges that must be removed so that the graph is disconnected into two parts each with an equal, within 1, number of vertices. The bisection width of the interconnection network of a parallel machine can be used in determining the time required to perform a computation by that machine [14], and the area needed to layout this network in Thompson's model [13]. However, the problem of determining the bisection width of a graph is NP–hard (reduction from MINIMUM CUT INTO BOUNDED SETS [8]).

The transposition network T_n is the Cayley graph of the symmetric group S_n with generators the set of all transpositions in S_n. In T_n, there is an edge from a vertex with label a permutation $\pi_1 \in S_n$ to a vertex with label a permutation $\pi_2 \in S_n$ iff $\pi_1^{-1} \cdot \pi_2$ is a transposition. Consequently, T_n is a regular undirected graph with degree $n(n-1)/2$ and with diameter $n-1$. Further, since each edge in T_n has an end-vertex labeled by an even permutation in S_n and the other end-vertex labeled by an odd permutation in S_n, T_n is a bipartite graph. Other combinatorial properties of T_n have been investigated in [11].

F. T. Leighton [14, p. 776] poses the problem of finding the bisection width of the transposition network T_n as an open problem. In this paper, we resolve this open problem for n even by showing that the bisection width of T_n is equal to $nn!/4$. When $n \geq 2$ is odd, we show that the bisection width of T_n is between $nn!/4$ and $(1+1/n-2/n^2)nn!/4$. In doing so, using the representation theory of the symmetric group, we determine that the second largest eigenvalue of the adjacency matrix of T_n, $n \geq 2$, is equal to $n(n-3)/2$. Using this fact, we also find that the isoperimetric number of T_n is equal to $n/2$, for n even, and is $(1+o(1))n/2$ otherwise, and conclude that the transposition network T_n can simulate, with $O(\log n)$ slowdown, any bounded degree network with $\leq n!$ nodes.

Further, given a Cayley graph of a finite group G with a set of generators closed under taking conjugates and not containing the identity of G, we show how to construct an $m \times m$ matrix Q, where m is the number of conjugacy classes of G, such that the set of eigenvalues of Q is equal to the set of eigenvalues of the adjacency matrix A of the given Cayley graph. Matrix Q is the adjacency matrix of a certain multi-graph, which is constructed by collapsing, for each conjugacy class of G, all conjugate vertices of this Cayley graph into a single vertex in this multi-graph. Curtis and Reiner [6, p. 234–235] essentially present a proof of a special case of this result, when this set of generators is a single conjugacy class of G. This result is important since when the order of G is large compared to the number of its conjugacy classes, we obtain a faster method for computing the eigenvalues of A by computing the eigenvalues of Q. For example, this approach can be followed for the transposition network T_n.

The rest of the paper is organized as follows. In section 2 we provide some preliminaries. In section 3 we provide an upper bound on the bisection width of T_n and we determine the edge and vertex connectivity of T_n. In section 4 we provide a lower bound on the bisection width of the transposition network T_n by determining the second largest eigenvalue of the adjacency matrix of T_n. In section 5 we provide an alternative characterization for the eigenvalues of the adjacency matrix of the Cayley graph of a group with set of generators closed under taking conjugates.

2 Preliminaries

We assume that the reader is familiar with basic graph theory and algebra. For graph theoretic terms or algebra terms that are used but not defined in this paper, see Berge [3] or Herstein [10], respectively.

A permutation π on $\{1, 2, \ldots, n\}$ is a bijection from $\{1, 2, \ldots, n\}$ onto itself. Given two permutations π_1, π_2 on $\{1, 2, 3, \ldots, n\}$, the product of π_1 by π_2, denoted by $\pi_1 \cdot \pi_2$, is a permutation π on the same set such that $\pi(i) = \pi_2(\pi_1(i))$, $i = 1, 2, \ldots, n$. The set of all permutations on $\{1, 2, 3, \ldots, n\}$ together with the operation of multiplication of permutations forms a group of order $n!$, called the *symmetric group of degree* n and denoted by S_n.

A cycle $(i_1, i_2, \ldots, i_k)$ of length k on S_n, where $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\}$, represents the permutation in S_n that maps i_1 to i_2, i_2 to i_3, …, i_{k-1} to i_k, i_k to i_1, and maps all other elements to themselves. A cycle in S_n of length 2 is called a *transposition*. A permutation is called an *even permutation* if it can be expressed as the product of an even number of transpositions, otherwise it is called an *odd permutation*. The number of even permutations in S_n is equal to $n!/2$.

Let G be a (multiplicative) finite group. Recall that a subset $C \subseteq G$ *generates* G if every element of G can be expressed as a product of elements from C. In this case, we call C a *set of generators* of G. Also recall that two elements g_1, g_2 of G are called *conjugate* if there exists an element $\theta \in G$ such that $\theta^{-1} g_1 \theta = g_2$. Conjugacy is an equivalence relation on G and the equivalence classes of this relation are called conjugacy classes of G. Let C be a set of generators for G. We construct a directed graph, called the *Cayley graph* of group G with set of generators C and denoted by $\mathrm{Cay}(G, C)$, as follows. For each element $a \in G$ we have a vertex, that is labeled with a, in $\mathrm{Cay}(G, C)$. There is an edge in $\mathrm{Cay}(G, C)$ from a vertex with label a_1 to a vertex with label a_2 iff there exists a generator $g \in C$ such that $a_1 \cdot g = a_2$, and in such a case we label that edge with g. It follows that all edges incident to any vertex of $\mathrm{Cay}(G, C)$ are labeled with distinct generators. A set $S \subseteq G$ is *closed under taking inverses* if for each $g \in G$, $g \in S$ implies that $g^{-1} \in S$. A set $S \subseteq G$ is *closed under taking conjugates* if for each $g \in G$, $g \in S$ implies that each conjugate $g' \in G$ of g is also in

S. Whenever the set of generators C is closed under inverses, the Cayley graph $\mathrm{Cay}(G, C)$ is an undirected graph.

A sorted sequence of positive integers $(n_1, n_2, \ldots, n_k)$, $n_1 \geq n_2 \geq \ldots \geq n_k$, for which $n = n_1 + n_2 + \ldots + n_k$, is called *a partition of* n. Let $p(n)$ denote the number of partitions of n. Whenever we express a permutation $\pi \in S_n$ as a product of disjoint cycles, including cycles of length 1 so that each integer in $\{1, 2, \ldots, n\}$ appears exactly once in that product, we obtain a partition $(n_1, n_2, \ldots, n_k)$ of n, called the *cycle decomposition* of π, where the cycles in that product have lengths $n_1, n_2, \ldots, n_k$. Two permutations in S_n are conjugate iff they have the same cycle decomposition. The number of conjugacy classes of S_n is equal to $p(n)$.

In the remainder of this section, we define the concepts of matrix representations of a group and the group characters they induce. For a detailed discussion of these concepts, we refer the interested reader to Curtis and Reiner [6]. Matrix representations and their induced characters are used in section 4.1.

A *matrix representation of a group* G *of degree* n *over a field* K is a homomorphism $T : G \rightarrow GL(n, K)$, where n is a positive integer and $GL(n, K)$ is the group of $n \times n$ invertible matrices over K. That is, to each element $g \in G$ there corresponds an $n \times n$ invertible matrix $T(g)$ over K. If e is the identity element of G then $T(e) = I$, $T(g^{-1}) = T(g)^{-1}$, and for all $g_1, g_2 \in G$, $T(g_1 \cdot g_2) = T(g_1)T(g_2)$. (See [6, p. 30].) Given a matrix representation T of a group G over a field K, the *(group) character* that T affords for G is a function $\chi : G \rightarrow K$ such that for each $g \in G$, $\chi(g) = \mathrm{tr}(T(g))$, where $\mathrm{tr}(T(g))$ is the trace of $T(g)$. The group characters are class functions, *i.e.* for any two conjugate elements $g_1, g_2 \in G$, $\chi(g_1) = \chi(g_2)$. A matrix representation T of a group G of degree n over a field K, that satisfies certain rather technical conditions (see [6, p. 40] for details), is called an *irreducible matrix representation of* G of degree n over K, where n is a positive integer. The group character afforded by this irreducible matrix representation T of G is called an *irreducible (group) character* of G. Hereafter, we assume that the field K is the field of complex numbers, and consequently we are dealing with complex matrix representations and their corresponding complex characters.

3 An Upper Bound on the Bisection Width

In this section, we obtain an upper bound on the bisection width of the transposition network T_n, $n \geq 2$, by providing a method to bisect T_n. In addition, we find that the vertex and edge connectivity of the transposition network T_n are both equal to its degree $n(n-1)/2$, *i.e.* T_n is maximally fault-tolerant.

First, we show the following.

Lemma 1 *The transposition network* T_n*,* $n \geq 2$*, can be partitioned into* n *vertex–disjoint* *subgraphs* $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$*, such that each subgraph* Γ_i *is isomorphic to* T_{n-1}*. Further, for each* $i, j = 1, 2, \ldots, n$*,* $i \neq j$*, each vertex in* Γ_i *is connected to exactly one vertex in* Γ_j*.*

Proof: Each of the vertices of T_n is labeled with a permutation in the symmetric group S_n. For each $i = 1, 2, \ldots, n$, let Γ_i be the induced subgraph of T_n that contains all the vertices labeled with a permutation π such that $\pi(n) = i$. Then, the subgraphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ constitute a partition of T_n into vertex–disjoint subgraphs. Moreover, for $i = 1, 2, \ldots, n$, by mapping each vertex of Γ_i, which is labeled with $\pi \in S_n$, to the vertex of T_{n-1}, that is labeled with the permutation $\pi \cdot (n, i)$ restricted to $\{1, 2, \ldots, n-1\}$, we find that the graph Γ_i is isomorphic to T_{n-1}. Consider now two distinct such subgraphs Γ_i and Γ_j. Consider a vertex v in Γ_i, which is labeled with $\pi \in S_n$. Vertex v is connected to a vertex u in T_n, that is labeled with $\pi' = \pi \cdot (i, j)$. Since $\pi'(n) = j$, vertex u is in Γ_j. Since the only transposition that we can multiply π with to get a permutation that maps n to j is the transposition (i, j), it follows that each vertex in Γ_i is connected to exactly one vertex in Γ_j. ■

Second, using Lemma 1, we show the following.

Lemma 2 *For* $n \geq 2$*, the bisection width of the transposition network* T_n *is* $\leq nn!/4$ *if* n *is even, and is* $\leq (1 + 1/n - 2/n^2)nn!/4$ *otherwise.*

Proof: We prove this lemma by induction on n.

Basis: $n = 2$. Since T_2 has two vertices and one edge, its bisection width is 1.

Inductive Hypothesis: Suppose that the lemma is true for T_k, $n > k \geq 2$.

Inductive Step: We show that the lemma holds for T_n, $n \geq 3$. First, suppose that n is even. Let X contain those vertices of T_n that correspond to $\Gamma_1, \Gamma_2, \ldots, \Gamma_{n/2}$. By Lemma 1 it follows that $|E(X, V(T_n) - X)| = nn!/4$, where $E(A, B)$ denotes the set of edges of T_n with one vertex in A and the other in B. Second, suppose that n is odd. Bisect $\Gamma_{(n+1)/2}$ as above. Let X' be the vertices in one part of that bisection. Let X consist of the vertices of T_n that correspond to $\Gamma_1, \Gamma_2, \ldots, \Gamma_{(n-1)/2}$ and the vertices in X'. By Lemma 1 and the inductive hypothesis for $\Gamma_{(n+1)/2}$ it follows that $|E(X, V(T_n) - X)| = (1 + 1/n - 2/n^2)nn!/4$. Consequently, the lemma holds for all integers $n \geq 2$. ■

Third, using Lemma 1, we determine that the vertex and edge connectivity of T_n are both equal to its degree $n(n-1)/2$. Since the vertex and edge connectivity are used to measure the fault–tolerance of a network, we conclude that the transposition network is maximally fault-tolerant with respect to vertex and edge connectivity.

Corollary 1 *The vertex and edge connectivity of the transposition network* T_n*,* $n \geq 2$*, are both equal to its degree* $n(n-1)/2$*.*

Proof: The vertex connectivity of T_n is trivially bounded from above by $n(n-1)/2$. By induction on n, we show that the vertex connectivity of T_n is equal to $n(n-1)/2$. The vertex connectivity of T_2 is equal to 1. Suppose that the vertex connectivity of T_{n-1} is equal to $(n-1)(n-2)/2$, $n \geq 3$. We show that we can not disconnect T_n by removing less than $n(n-1)/2$ vertices. Suppose that we remove $< n(n-1)/2$ vertices from T_n. Let T'_n be the induced graph after removing those vertices, and let Γ'_i be the subgraph of T'_n that is induced from Γ_i, $i = 1, 2, \ldots, n$. By Lemma 1 it follows that if $\Gamma'_i = \Gamma_i$ then T'_n is connected. Thus, suppose that at least one vertex is removed from each Γ_i. Then, less than $n(n-1)/2 - n + 1 = (n-1)(n-2)/2$ vertices are removed from each Γ_i. By Lemma 1 and the inductive hypothesis for Γ_i it follows that each Γ'_i is connected. Moreover, there exists a vertex in Γ'_i that is connected to a vertex in Γ'_j, $i, j = 1, 2, \ldots, n$. Consequently, T'_n is connected and the vertex connectivity of T_n is equal to $n(n-1)/2$. Since the vertex connectivity is a lower bound on the edge connectivity of T_n, and the edge connectivity is no more than the degree of T_n, the edge connectivity of T_n is also equal to $n(n-1)/2$. ■

4 A Lower Bound on the Bisection Width

We show that the bisection width of the transposition network T_n, $n \geq 2$, is bounded from below by $nn!/4$. To this end, we show that the second largest eigenvalue of the adjacency matrix of T_n is equal to $n(n-3)/2$. Then, using Proposition 2.1 in [16], we obtain this $nn!/4$ lower bound on the bisection width of T_n.

4.1 Expressions for the Eigenvalues

A theorem by Zieschang [19] and certain expressions for the irreducible characters of the symmetric group S_n over the field of complex numbers provide expressions for the eigenvalues of the adjacency matrix of the transposition network.

Theorem 1 (Zieschang) *Let G a finite group with m conjugacy classes, where m is a positive integer. Let C be a set of generators for G that is closed under taking conjugates and does not contain the identity g_e of G. Let A be the adjacency matrix of the Cayley graph* $\mathrm{Cay}(G, C)$. *Then, the eigenvalues of A are*

$$\mathrm{Eig}(A) = \{ \sum_{g \in C} \chi_i(g)/\chi_i(g_e) \ : \ i = 1, 2, \ldots, m\} \tag{1}$$

and the multiplicity of $\lambda \in \mathrm{Eig}(A)$ as an eigenvalue of A is equal to

$$\sum_{\lambda = \sum_{g \in C} \chi_i(g)/\chi_i(g_e)} \chi_i(g_e)^2. \tag{2}$$

where $\chi_i : G \rightarrow \mathcal{C}$, $i = 1, 2, \ldots, m$, are the irreducible complex characters of G, and $\mathcal{C}$ denotes the set of complex numbers.

Proof: See Zieschang [19]. See also Curtis and Reiner [6, p. 234–235]. ■

The eigenvalues of the adjacency matrix A_n of the transposition network T_n can be determined using Theorem 1. To this end, we are using the formulas for the irreducible complex character $\chi_i(\tau)$ of a transposition $\tau \in S_n$ and for the irreducible complex character of the identity permutation $\pi_e \in S_n$ provided in [18]. (It is known that the values of the irreducible characters of the symmetric group are all rational numbers.)

The irreducible complex characters $\chi_i(\pi_e)$ and $\chi_i(\tau)$, $i = 1, 2, \ldots, p(n)$, are determined with respect to partitions of n. There is a 1–1 correspondence between the irreducible complex characters of S_n and the partitions of n [6, 18]. Let the partition of n that corresponds to an irreducible complex character χ_i be $(n_1, n_2, \ldots, n_k)$, $k \geq 1$. [4] Then, from equation (5.6) in [18, p. 140], it follows that

$$\chi_i(\tau)/\chi_i(\pi_e) = \sum_{j=1}^{k} \frac{n_j(n_j - 2j + 1)}{n(n-1)}, \tag{3}$$

where τ is a transposition in S_n and π_e is the identity permutation in S_n. As for $\chi_i(\pi_e)$, it is known that this can determined using the Young Diagram associated with the partition $(n_1, n_2, \ldots, n_k)$ of n and a corresponding theorem by Frame, Robinson, and Thrall [2, p. 63]. The *Young diagram* [2, p. 49] associated with a partition $(n_1, n_2, \ldots, n_k)$ of n is a diagram that consists of k rows and n_1 columns of boxes, where row l contains n_l boxes. Each box holds a number in $\{1, 2, \ldots, n\}$ and no two boxes hold the same number. A *hook with corner at* (l, j) is a Γ shaped figure of boxes that consists of all the boxes in row l and in columns $j, j+1, \ldots, n_1$, and all the boxes in column j and in rows $l, l+1, \ldots, k$. The length of the hook with corner at (l, j), denoted by $h_{l,j}$, is equal to the number of boxes it contains. Then, from a theorem by Frame, Robinson, and Thrall [2, p. 63], it follows that

$$\chi_i(\pi_e) = \frac{n!}{\Pi_{l=1}^{k} \Pi_{j=1}^{n_1} h_{l,j}}. \tag{4}$$

(See also [18, Equation 4.34, p. 119].)

Consequently, given all the partitions of n and using equations (1), (2), (3), and (4), we can find all the eigenvalues of A_n together with their multiplicities. Recall that the number of partitions $p(n)$ of n is a fast growing function, $p(n) = \Theta(\exp(\pi\sqrt{2n/3} - \log(4\sqrt{3}n)))$ [2, p. 56].

[4] Recall that $(n_1, n_2, \ldots, n_k)$ is a non-increasing sequence.

4.2 The Second Largest Eigenvalue

Since $p(n)$ is a fast growing function of n, closed form expressions for the eigenvalues of A_n together with their multiplicities are useful. In particular, a closed form expression for the second largest eigenvalue of A_n will allow us to compute among other things a lower bound on the bisection width of T_n.

Lemma 3 *Let $n \geq 2$ be an integer. Then, all the eigenvalues of the adjacency matrix A_n of the transposition network T_n are integers, and the algebraic and geometric multiplicity of each one of them are equal. If λ is an eigenvalue of A_n then $-\lambda$ is also an eigenvalue of A_n with the same multiplicity as λ. The largest eigenvalue of A_n is $n(n-1)/2$ with multiplicity 1. The second largest eigenvalue of A_n is $n(n-3)/2$ with multiplicity $(n-1)^2$. For each $k = 1, 2, \ldots, n$, $n(n-2k+1)/2$ is an eigenvalue of A_n with multiplicity at least $n!/(n(n-k)!(k-1)!)$.*

Proof: We first argue that all the eigenvalues of A_n are integers. The characteristic polynomial $\phi_{n!}(x) = \det(xI - A_n)$ of A_n is a primitive polynomial of degree $n!$ with integer coefficients. [5] From equations (1) and (3) it follows that all the eigenvalues of A_n, which are also roots of $\phi_{n!}(x)$, are rational numbers. Since all the rational roots of a primitive polynomial with integer coefficients are indeed integers, all the eigenvalues of A_n are integers (see [10, Theorem 3.10.1]).

Because A_n is a symmetric matrix, it follows that the algebraic and geometric multiplicities of each eigenvalue of A_n are equal (see [7, Theorem 0.1]).

It is known that the spectrum of an undirected bipartite graph with at least one edge is symmetric with respect to 0 (see [7, Theorem 3.11]). Hence, since T_n, $n \geq 2$, is a bipartite graph with at least one edge, it follows that if λ is an eigenvalue of A_n then $-\lambda$ is also an eigenvalue of A_n with the same multiplicity as λ.

Since T_n is a connected regular graph with degree $n(n-1)/2$, we conclude from Proposition 3.1 in [5], that $n(n-1)/2$ is the largest eigenvalue of A_n and its multiplicity is 1. In addition, $-n(n-1)/2$ is the smallest eigenvalue of A_n and its multiplicity is 1. On the other hand, from (1), (2), (3), and (4), we find that the partition (n) of n gives rise to the eigenvalue $n(n-1)/2$ of A_n. Further, since $\chi_i(\pi_e)$, $i = 1, 2, \ldots, p(n)$, are the degrees of irreducible matrix representations of S_n, $\chi_i(\pi_e)$, $i = 1, 2, \ldots, p(n)$, are positive integers. Hence, only the partition (n) of n gives rise to the largest eigenvalue of A_n. Observe that $n(n-1)/2$ is the maximum value of

$$\sum_{j=1}^{k} \frac{n_j(n_j - 2j + 1)}{2} \tag{5}$$

[5] Recall that a polynomial $\phi_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ with integer coefficients is called primitive if the greatest common divisor of $a_0, a_1, a_2, \ldots, a_n$ is 1 [10, p. 159].

over all partitions $(n_1, n_2, \ldots, n_k)$ of n, $k \geq 1$.

We next show that the second largest eigenvalue of A_n is equal to $n(n-3)/2$ and has multiplicity $(n-1)^2$.

From (1) and (3), it follows that the second largest eigenvalue of A_n is equal to the maximum value of

$$\sum_{j=1}^{k} \frac{n_j(n_j - 2j + 1)}{2} \tag{6}$$

over all partitions $(n_1, n_2, \ldots, n_k)$ of n such that $k \geq 2$. Observe that $n_k \leq n/2$, since $n_1 \geq n_2 \geq \ldots \geq n_{k-1} \geq n_k \geq 1$ and $k \geq 2$. By taking the partition $(n-1, 1)$ of n, it follows from (6), (2), and (4) that $n(n-3)/2$ is an eigenvalue of A_n with multiplicity at least $(n-1)^2$. Thus, the second largest eigenvalue of A_n is no less than $n(n-3)/2$. Next, we find an upper bound on the value of the expression in (6). Since

$$\sum_{j=1}^{k} \frac{n_j(n_j - 2j + 1)}{2} = \sum_{j=1}^{k-1} \frac{n_j(n_j - 2j + 1)}{2} + \frac{n_k(n_k - 2k + 1)}{2} \tag{7}$$

and since $\sum_{j=1}^{k-1} n_j(n_j - 2j + 1)/2$ is bounded from above by $(n - n_k)(n - n_k - 1)/2$, we conclude that

$$\sum_{j=1}^{k} \frac{n_j(n_j - 2j + 1)}{2} \leq \frac{(n - n_k)(n - n_k - 1)}{2} + \frac{n_k(n_k - 2k + 1)}{2}. \tag{8}$$

Since $k \geq 2$ and $1 \leq n_k \leq n/2$, from (8) we have that

$$\frac{(n - n_k)(n - n_k - 1)}{2} + \frac{n_k(n_k - 2k + 1)}{2} \leq \frac{n(n-3)}{2}. \tag{9}$$

Therefore, $n(n-3)/2$ is the second largest eigenvalue of A_n. In addition, since there exists only one partition of $n - n_k$, namely $(n - n_k)$, for which $\sum_{j=1}^{k-1} n_j(n_j - 2j + 1)/2$ attains its maximum value $(n - n_k)(n - n_k - 1)/2$, and since $1 \leq n_k \leq n/2$ and $k \geq 2$, it follows that the only partition of n that gives rise to $n(n-3)/2$ as an eigenvalue of A_n is $(n-1, 1)$. Consequently, the multiplicity of $n(n-3)/2$ as an eigenvalue of A_n is equal to $(n-1)^2$.

Observe that, for each $k = 1, 2, \ldots, n$, the partition $(n-k+1, 1, 1, 1, \ldots, 1)$ of n furnishes $n(n-2k+1)/2$ as an eigenvalue of A_n with multiplicity at least $n!/(n(n-k)!(k-1)!)$. Note that, since the diameter of T_n is $n-1$, A_n has at least n distinct eigenvalues (see [5, Corollary 2.7]). ∎

4.3 The Bisection Width of the Transposition Network

The *Laplacian matrix* L of graph Γ is the matrix $L = D - A$, where A is the adjacency matrix of Γ and D is the diagonal matrix with the degrees of the vertices of Γ on the diagonal. Mohar [16, Proposition 2.1] asserts the following.

Proposition 1 (Mohar) *Let $\Gamma = (V, E)$ be a graph with n vertices. Let λ_2 and λ_n be the second smallest and largest eigenvalues of its Laplacian matrix, respectively. Then, $\lambda_2/n \leq |E(X, V - X)|/(|X||V - X|) \leq \lambda_n/n$ for any non-empty proper subset $X \subset V$.*

If Γ is a regular graph then λ_2 is equal to the difference of the degree of Γ and its second largest eigenvalue. Therefore, the second smallest eigenvalue of the Laplacian of T_n is equal to n, and the above mentioned bound implies the following.

Lemma 4 *For $n \geq 2$, the bisection width of the transposition network T_n is $\geq nn!/4$.*

Combining Lemmas 2 and 4, we conclude the following.

Theorem 2 *For $n \geq 2$, the bisection width of the transposition network T_n is equal to $nn!/4$ if n is even, and is between $nn!/4$ and $(1 + 1/n - 2/n^2)nn!/4$ otherwise.*

When n is odd, we estimate the bisection width of T_n within a $1 + o(1)$ factor.

Next, using the second smallest eigenvalue of the Laplacian matrix of the transposition network we compute upper and lower bounds on the isoperimetric number of T_n. The *isoperimetric number* (or *minimum edge expansion* or *flux*) $\iota(\Gamma)$ of a graph $\Gamma = (V, E)$ is

$$\iota(\Gamma) = \min\{\ |E(V', V - V')|/|V'|\ :\ V' \subseteq V \text{ and } 0 < |V'| \leq |V|/2\}, \tag{10}$$

where $E(V', V - V')$ denotes the set of edges of Γ that have one endpoint in V' and the other in $V - V'$. Computing the isoperimetric number of a graph seems a computationally difficult problem (see [12, 15, 17] and references therein). [6] Graphs with large isoperimetric number tend to grow rapidly and can be used in the explicit construction of expanders and superconcetrators. We provide upper and lower bounds on the isoperimetric number $\iota(T_n)$ of the transposition network T_n. It is known that the isoperimetric number of a graph Γ is $\geq \lambda_2/2$, where λ_2 is the second smallest eigenvalue of the Laplacian matrix of Γ [17, Eq. 21]. Thus, $\iota(T_n) \geq n/2$. On the other hand, by Lemma 2 it follows that $\iota(T_n) \leq n/2$ if n is even, and $\iota(T_n) \leq (1 + 1/n - 2/n^2)n/2$ otherwise. Hence, we have the following.

Theorem 3 *For $n \geq 2$, the isoperimetric number of the transposition network T_n is equal to $n/2$ if n is even, and is between $n/2$ and $(1 + 1/n - 2/n^2)n/2$ otherwise.*

[6] The problem of computing the isoperimetric number of a multi–graph is NP–hard [15]. To the best of out knowledge, the complexity of this problem is open for simple graphs.

Moreover, using Theorem 2 of Leighton and Rao [12] and these bounds on $\iota(T_n)$, we conclude that any bounded degree graph can be embedded in T_n with load 1, and simultaneous congestion and dilation $O(\log n)$, which further implies that the transposition network T_n can simulate any bounded degree network with $\leq n!$ nodes with $O(\log n)$ slowdown.

5 Divisors of Cayley Graphs

In this section, we show that the set of eigenvalues of the adjacency matrix of the Cayley graph of a finite group with a set of generators closed under taking conjugates and not containing the identity of G, is equal to the set of eigenvalues of a certain integer matrix, whose dimension is equal to the number of conjugacy classes of this group. A proof of a special case of this result, namely when this set of generators is equal to a single conjugacy class of this group, is essentially given in Curtis and Reiner [6, p. 235].

Let $\Gamma = (V, E)$ be a connected graph with adjacency matrix A. A partition $V_1, V_2, \ldots, V_k$ of V, $k \geq 1$, is called an *equitable partition* of Γ if for each $i, j = 1, 2, \ldots, k$ there exists a non-negative integer $d_{i,j}$ such that each vertex $v \in V_i$ has exactly $d_{i,j}$ neighbors that are in V_j. Let D be the $k \times k$ matrix whose (i, j) entry is equal to $d_{i,j}$ above. The multi-graph Γ_D whose adjacency matrix is D is called a *divisor* of Γ. It is known that $\mathrm{Eig}(D) \subseteq \mathrm{Eig}(A)$ [7, p. 123], where $\mathrm{Eig}(X)$ denotes the set of eigenvalues of a matrix X. As a consequence, equitable partitions of a graph are useful in factoring the characteristic polynomial of the adjacency matrix of that graph. Godsil and McKay [9, Corollary 4.10] prove that if Γ is *walk–regular*, that is the number of closed walks of length $k \geq 0$ that start at a vertex $u \in V$ is independent of u for any non–negative integer k, and if for any vertex $u \in V$ there exists an equitable partition π_u of Γ that has a block containing vertex u only, then $\mathrm{Eig}(A) = \mathrm{Eig}(D_u)$ for any $u \in V$, where D_u is the matrix that corresponds to π_u.

Observe that if a walk–regular graph $\Gamma = (V, E)$ is in addition vertex–symmetric, that is for any two vertices $u, v \in V$ there is an automorphism of Γ that maps u to v, one can apply this result of Godsil and McKay [9, Corollary 4.10] by finding any equitable partition of Γ that has a singleton block. This observation is particularly useful for Cayley graphs of finite groups, since those graphs are both walk–regular and vertex–symmetric.

Let G be a finite group and let $C_1, C_2, \ldots, C_m$ be the conjugacy classes of G. Let g_1 be the identity of G and let C_1 be the conjugacy class of G that contains g_1. Let A be the adjacency matrix of the Cayley graph $\mathrm{Cay}(G, C)$ of G with set of generators C closed under taking conjugates. Let $\mathcal{I} \subseteq \{2, 3, \ldots, m\}$ such that $C = \cup_{i \in \mathcal{I}} C_i$. It is not difficult to show, using standard group–theoretic arguments, that for any two conjugate classes C_i and C_j of G, there exists a non-negative integer $q_{i,j}$ such that the number of neighbors in $\mathrm{Cay}(G, C)$ of any vertex $g \in C_i$ that are in C_j is equal to $q_{i,j}$. Let Q be the $m \times m$ matrix of integers whose $(i, j)th$ entry is equal to $q_{i,j}$. Therefore, by partitioning the vertices of $\mathrm{Cay}(G, C)$ according to the conjugacy classes of G, we obtain an equitable partition of $\mathrm{Cay}(G, C)$. We define the *conjugacy divisor* of Cay(G, C) to be the multi-graph whose adjacency matrix is Q. Note that this equitable partition of Cay(G, C) has a block containing a single vertex only. Then, since Cay(G, C) is walk–regular and vertex–symmetric, we have the following.

Theorem 4 *Let A be the adjacency matrix of a Cayley graph* Cay(G, C) *of a group G with a set of generators C that is closed under taking conjugates and does not contain the identity of G. Let Q be the adjacency matrix of the conjugacy divisor of* Cay(G, C). *Then,* Eig(A) = Eig(Q).

Proof: Follows from the discussion above and Corollary 4.10 in Godsil and McKay [9](as suggested by one of the referees). [7] ■

Whenever the number of conjugacy classes of G is small compared to the order of G, Theorem 4 provides a more efficient method for computing the eigenvalues of A, up to multiplicity, by computing the eigenvalues of the smaller matrix Q. In particular, Theorem 4 can be used to find all the eigenvalues, up to multiplicity, of the adjacency matrix A_n of the transposition network T_n, where A_n is an $n! \times n!$ matrix, by computing all the eigenvalues of a $p(n) \times p(n)$ integer matrix, where $p(n)$ is the number of partitions of n.

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