CMSC 341

Disjoint Sets

Textbook Chapter 8

Equivalence Relations

- A relation R is defined on a set S if for every pair of elements (a, b) with $a,b \in \mathrm{S}$, $a\ R\ b$ is either true or false. If $a\ R\ b$ is true, we say that "a is related to b".
- An equivalence relation is a relation R that satisfies three properties
  - (Reflexive) $a\ R\ a$ for all $a \in \mathrm{S}$
  - (Symmetric) $a\ R\ b$ if and only if $b\ R\ a$
  - (Transitive) $a\ R\ b$ and $b\ R\ c$ implies that $a\ R\ c$

Equivalence Relation Examples

- =, but not ≤
- Students with the same eye color
- All cities in the same country
- Computers connected in a network

Equivalence Classes

- The equivalence class for an element $a \in S$ is the subset of S that contains all the elements that are related to a.
- The subsets that represent the equivalence classes will be “disjoint”
- Example
  - All students in CMSC 341 who are juniors

Equivalence Relation Application

- **Suppose we have an application involving N distinct items. We will not be adding new items, nor deleting any items. Our application requires us to use an equivalence relation to partition the items into a collection of equivalence classes (subsets) such that:**
  - each item is in a set,
  - no item is in more than one set.
- **Examples**
  - Classify UMBC students according to class rank.
  - Classify CMSC 341 students according to GPA.

Disjoint Set Terminology

- We identify a set by choosing a ***representative element*** of the set. It doesn’t matter which element we choose, but once chosen, it can’t change.
- There are two operations of interest:
  - find (x) -- determine which set x is in. The return value is the representative element of that set
  - union (x, y) -- make one set out of the sets containing x and y.
- Disjoint set algorithms are sometimes called ***union-find*** algorithms.

Disjoint Set Example

Given a set of cities, C, and a set of roads, R, that connect two cities (x, y) determine if it's possible to travel from any given city to another given city.

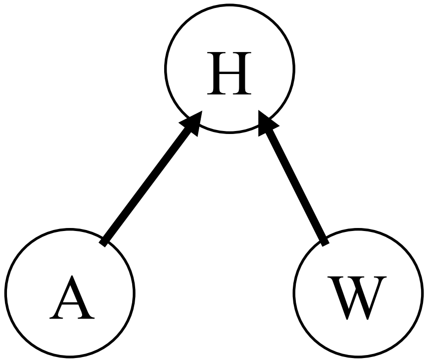
```
for (each city in C)
    put each city in its own set
for (each road (x,y) in R)
    if (find( x ) != find( y ))
        union(x, y)
```

Now we can determine if it's possible to travel by road between two cities c_1 and c_2 by testing

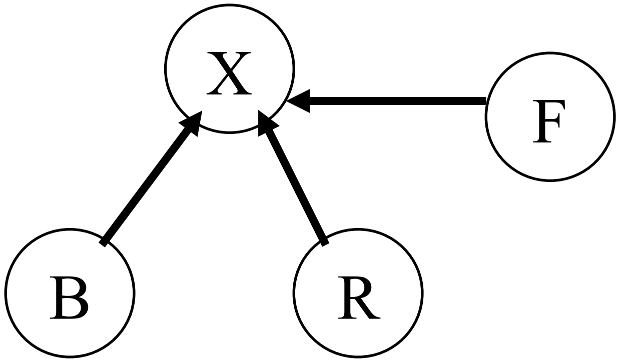
```
    find(c1) == find(c2)
```

Up-Trees

- A simple data structure for implementing disjoint sets is the *up-tree*.



H, A and W belong to the same set. H is the representative.



X, B, R and F are in the same set. X is the representative.

Operations in Up-Trees

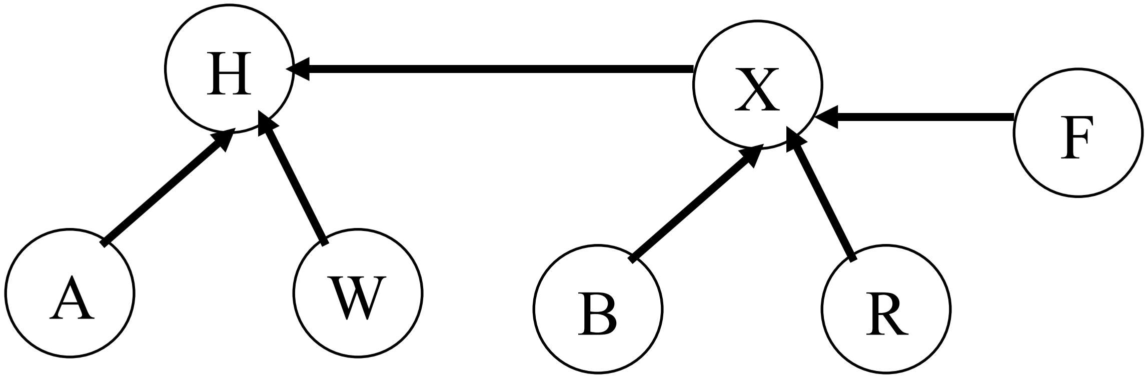
find( ) is easy. Just follow pointer to representative element. The representative has no parent.

```
find(x)
{
   if (parent(x)) // not the representative
                    return(find(parent(x));
   else
        return (x); // representative
}
```

Union

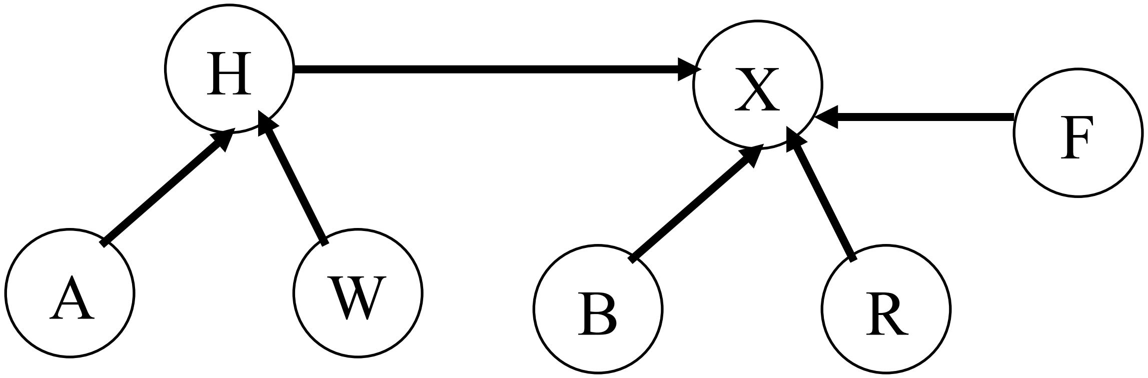
- Union is more complicated.
- Make one representative element point to the other, but which way?
  Does it matter?
- In the example, some elements are now twice as deep as they were before.

Union(H, X)



X points to H.

B, R and F are now deeper.

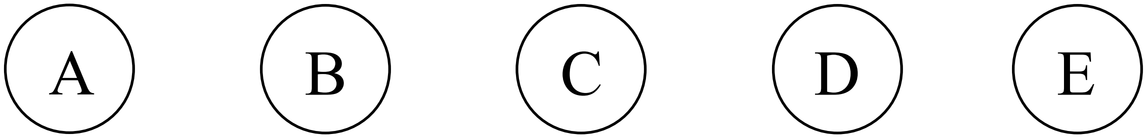


H points to X.

A and W are now deeper.

A Worse Case for Union

Union can be done in O(1), but may cause find to become O(n).



Consider the result of the following sequence of operations:

Union (A, B)
Union (C, A)
Union (D, C)
Union (E, D)

Array Representation of Up-tree

- Assume each element is associated with an integer i = 0…n-1. From now on, we deal only with i.
- Create an integer array, s[n]
- An array entry is the element’s parent
- s[i] = -1 signifies that element i is the representative element.

Union/Find with an Array

Now the union algorithm might be:

```
public void union(int root1, int root2) {
    s[root2 ] = root1; // attaches root2 to root1
}
```

The find algorithm would be

```
public int find(int x) {
    if (s[x ] < 0)
            return(x);
    else
            return(find(s[x ]));
}
```

Improving Performance

- There are two heuristics that improve the performance of union-find.
  - Path compression on find
  - Union by weight

Path Compression

Each time we find( ) an element E, we make all elements on the path from E to the root be immediate children of root by making each element's parent be the representative.

```
public int find(int x) {
    if (s[x ]<0)
          return(x);
    s[x ] = find(s[x ]);  // new code
    return (s[x ]);
}
```

When path compression is used, a sequence of m operations takes O(m lg n) time. Amortized time is O(lg n) per operation.

“Union by Weight” Heuristic

Always attach the smaller tree to larger tree.

```
public void union(int root1, int root2) {
   rep_root1 = find(root1);
   rep_root2 = find(root2);
   if(weight[rep_root1 ] < weight[rep_root2 ]){
        s[rep_root1 ] = rep_root2;
        weight[rep_root2 ]+= weight[rep_root1 ];
   }
   else {
        s[rep_root2 ] = rep_root1;
        weight[rep_root1 ] += weight[rep_root2 ];
   }
}
```

Performance with Union by Weight

- If unions are performed by weight, the depth of any element is never greater than lg N.
- Intuitive Proof:
  - Initially, every element is at depth zero.
  - An element's depth only increases as a result of a union operation if it's in the smaller tree in which case it is placed in a tree that becomes at least twice as large as before (union of two equal size trees).
  - Only lg N such unions can be performed until all elements are in the same tree
- Therefore, find( ) becomes O(lg n) when union by weight is used -- even without path compression.

Performance with Both Optimizations

- When both optimizations are performed a sequence of m ($m \geq n$) operations (unions and finds), takes no more than O(m lg* n) time.
  - lg*n is the iterated (base 2) logarithm of n -- the number of times you take lg n before n becomes ≤ 1.

- Union-find is essentially O(m) for a sequence of m operations (amortized O(1)).

A Union-Find Application

- A random maze generator can use union-find. Consider a 5x5 maze:

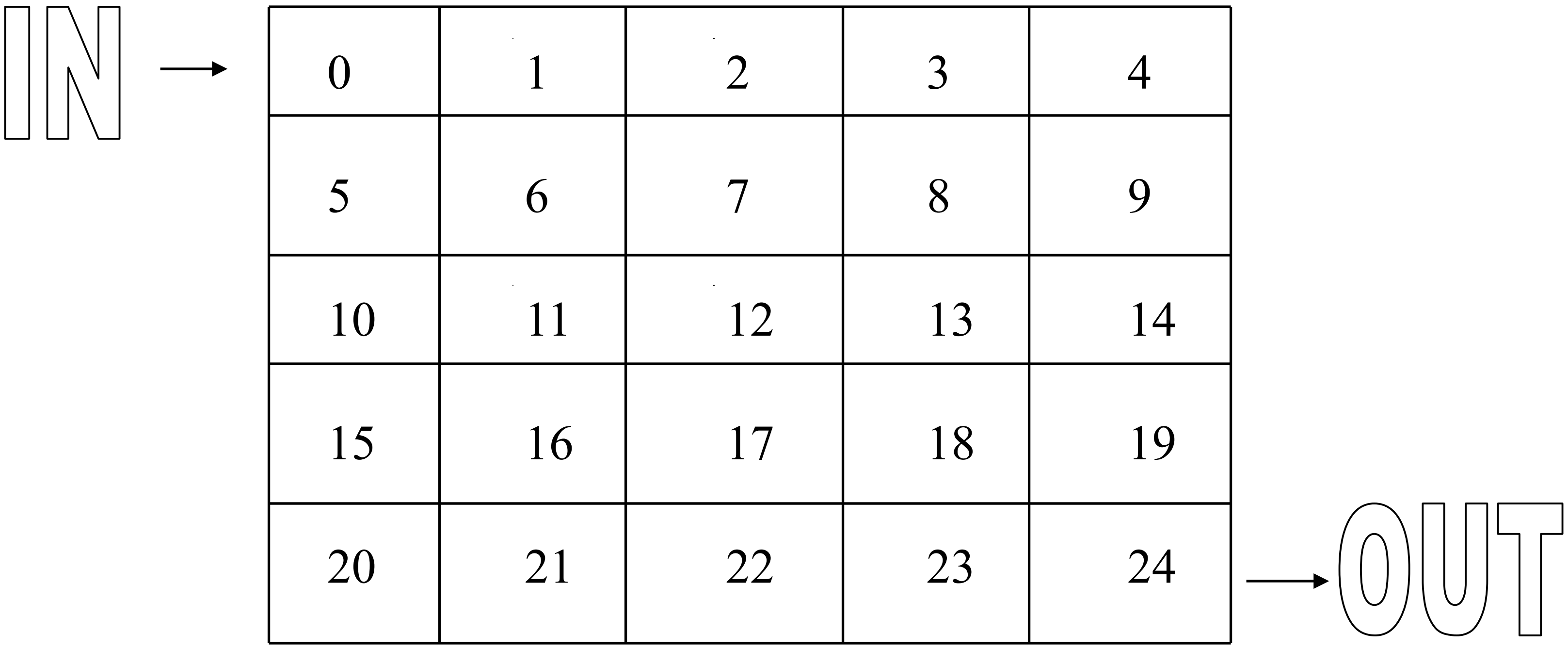
0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24

Maze Generator

- Initially, 25 cells, each isolated by walls from the others.
- This corresponds to an equivalence relation -- two cells are equivalent if they can be reached from each other (walls been removed so there is a path from one to the other).

Maze Generator (cont.)

- To start, choose an entrance and an exit.



0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24

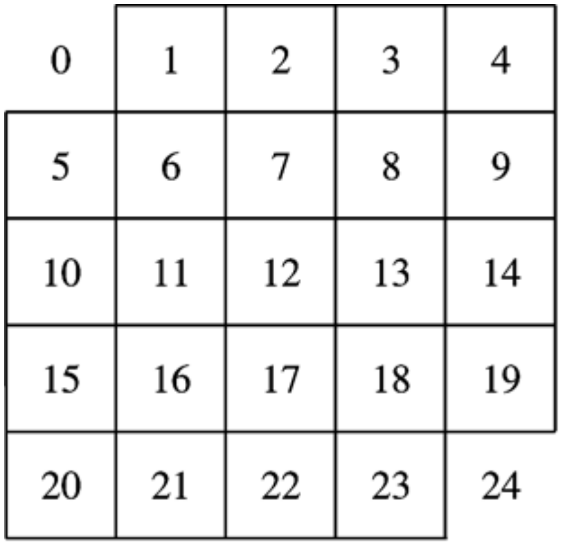
Maze Generator (cont.)

- Randomly remove walls until the entrance and exit cells are in the same set.
- Removing a wall is the same as doing a union operation.
- Do not remove a randomly chosen wall if the cells it separates are already in the same set.

MakeMaze

```
MakeMaze(int size) {
   entrance = 0; exit = size-1;
   while (find(entrance) != find(exit)) {
        cell1 = a randomly chosen cell
        cell2 = a randomly chosen adjacent cell
        if (find(cell1) != find(cell2)
                union(cell1, cell2)
   }
}
```

Initial State

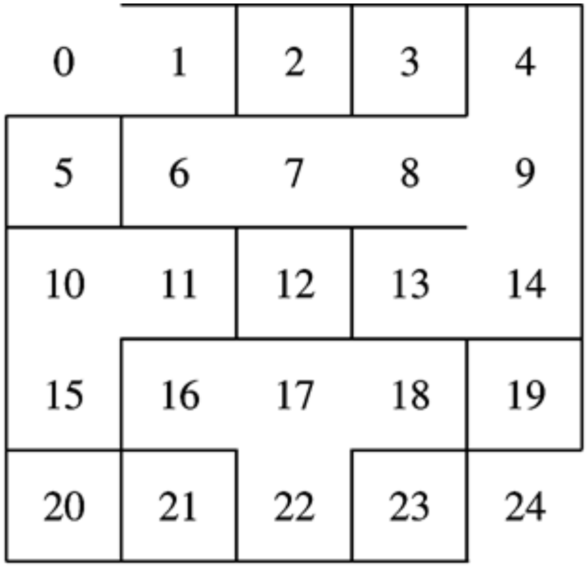


0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24

{0} {1} {2} {3} {4} {5} {6} {7} {8} {9} {10} {11} {12} {13} {14} {15} {16} {17} {18} {19} {20} {21} {22} {23} {24}

Intermediate State

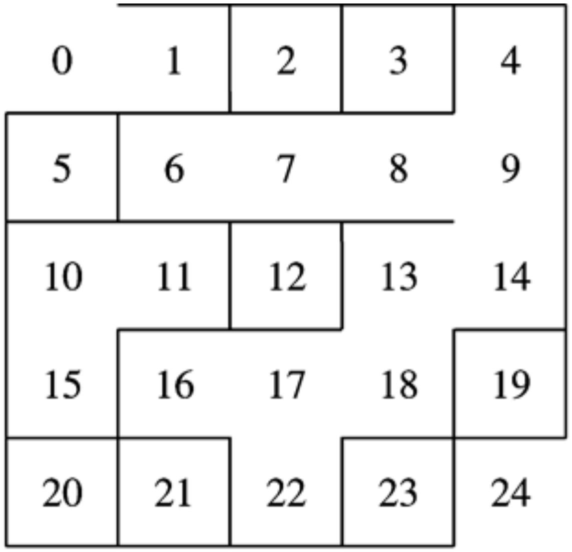
- Algorithm selects wall between 18 and 13. What happens?



{0, 1} {2} {3} {4, 6, 7, 8, 9, 13, 14} {5} {10, 11, 15} {12} {16, 17, 18, 22} {19} {20} {21} {23} {24}

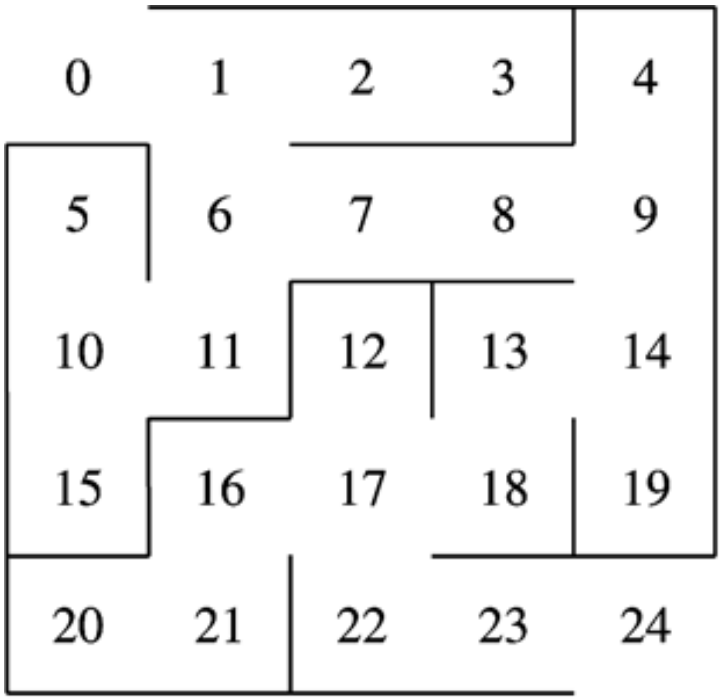
A Different Intermediate State

- Algorithm selects wall between 8 and 13. What happens?



{0, 1} {2} {3} {4, 6, 7, 8, 9, 13, 14, 16, 17, 18, 22} {5} {10, 11, 15} {12} {19} {20} {21} {23} {24}

Final State



{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24}